ISSN: 1392-6292/ eISSN: 1648-3510

MATHEMATICAL MODELLING and ANALYSIS

2025
Volume 30
Issue 1
Pages 169–185
https://doi.org/10.3846/mma.2025.20017

Discrete Gronwall's inequality for Ulam stability of delay fractional difference equations

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Article History:
- received September 26, 2023
- revised March 6, 2024
- accepted April 2, 2024

Abstract. This paper investigates Ulam stability of delay fractional difference equations. First, a useful equality of double fractional sums is employed and discrete Gronwall's inequality of delay type is provided. A delay discrete-time Mittag-Leffler function is used and its non-negativity condition is given. With the solutions' existences, Ulam stability condition is presented to discuss the error estimation of exact and approximate solutions.

Keywords: delay fractional difference equations; discrete Gronwall's inequality; uniqueness of solution; Ulam stability.

AMS Subject Classification: 26A33; 37H10; 39A05.

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1 Introduction

In 1940, Ulam [24] proposed a problem related to the stability of group homomorphism: Let G_1 be a group and G_2 is a metric group. For a given $\varepsilon > 0$, there exists $\delta > 0$ such that for any x and $y \in G_1$, the mapping p satisfying an inequality

$$d(p(xy), p(x)p(y)) < \delta,$$

whether there is a homomorphism $T : G_1 \to G_2$, for all $x \in G$, it satisfies

$$d(p(x), T(x)) < \varepsilon?$$

If the answer to this question is yes, it is said that the functional equation corresponding to the homomorphism $T(xy) = T(x)T(y)$ is stable.

One year later, Hyers [18] gave an answer to the Ulam problem. He extended the stability of group homomorphisms to the stability in Banach spaces. Let $p : E_1 \to E_2$ be a mapping between the Banach spaces E_1 and E_2, if

$$\|p(x + y) - p(x) - p(y)\| \leq \varepsilon \quad (x, y \in E_1),$$

where $\varepsilon > 0$ is a constant. Then for any $x \in E_1$, the limit

$$T(x) = \lim_{n\to\infty} 2^{-n} p\left(2^n x\right)$$

exists, and T is the only class additive mapping that satisfies

$$\|p(x) - T(x)\| \leq \varepsilon.$$

During 1978-1988, Rassias established the Hyers-Ulam stability theory of linear and nonlinear mappings [22]. In recent years, many researchers have paid attention to Ulam stability and obtained rich results. Rezaei used Laplace transform method to study the Hyers-Ulam stability of linear differential equations [23]. Wang used the Gronwall's inequality [28] to discuss Ulam stability and data dependence of fractional differential equation [25]. It plays an important role in theories of fractional differential equations.

Fractional difference equations become popular very recently [1,4,15], for example, boundary value problem [13], right fractional difference equations [27], neural networks [16], aftershock modeling [19], interval-valued system [17], and Laplace transform [6]. Due to extensive applications, the Gronwall's inequality was proposed in [26] and paid much attention by other researchers.

Discrete Gronwall's inequalities without delay were given in [5,14,26] for the fractional difference equation

$$^{C}\Delta_a^{\nu} x(t) = g(t+\nu)x(t+\nu),\ t \in \mathbb{N}_{a+1-\nu}. \tag{1.1}$$

Alzabut et al. [3], Du et al. [11] and Chen et al. [8] also gave serval other Gronwall's inequalities in the discrete fractional calculus.

The fractional difference equation

$$^{C}\Delta_a^{\nu} x(t) = g(t+\nu-1)x(t+\nu-1),\ t \in \mathbb{N}_{a+1-\nu}, \tag{1.2}$$

has one delay term as $g(t+\nu-1)x(t+\nu-1)$. Equations (1.1) and (1.2) are totally different. The fractional nonlinear difference equation of delay type has rich nonlinear dynamics in [16] and Equation (1.2) can be considered as the linearized version.

Gronwall's inequality is the fundamental one in mathematical modelling and analysis. This paper investigates the following fractional sum inequality

$$x_{n+1} \leqslant f_n + \frac{1}{\Gamma(\nu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} g_j x_j,\ 0 \leq n,\ 0 < \nu \leq 1,$$

where f_n and g_n are non-negative and non-decreasing functions.

The rest of the paper is organized as follows. In Section 2, the concept and some properties of discrete fractional calculus are explained. In Section 3, the non-negativity condition of Mittag-Leffler function is given. In Section 4, the discrete Gronwall's inequality of delay type is given. In Section 5, the uniqueness of fractional difference equations are discussed by using Banach fixed point theorem and Ulam stability of fractional difference equations is studied.

2 Preliminaries

Suppose $\mathbb{N}_a := \{a, a+1, \ldots\}, a \in \mathbb{R}$. For any $\nu \in \mathbb{R}$, the falling factorial functional is defined by [15]

$$t^{(\nu)} = \Gamma(t+1)/\Gamma(t+1-\nu),\ t \in \mathbb{N}_\nu,$$

where Γ denotes the famous Gamma function.
We use the following definitions in this paper.

Definition 1. [15] Let $x : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given. Then the ν-th order fractional sum of x is given by

$$\Delta_a^{-\nu} x(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} x(s),\ t \in \mathbb{N}_{a+\nu},$$

where $\sigma(s) = s + 1$.

Definition 2. [15] Let $x : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ be given, and $N - 1 < \nu \leq N$. Then the ν-th order Riemann-Liouville difference of x is given by

$$\Delta_a^{\nu} x(t) = \begin{cases} \dfrac{1}{\Gamma(-\nu)} \displaystyle\sum_{s=a}^{t+\nu} (t - \sigma(s))^{(-\nu-1)} x(s), & N - 1 < \nu < N, \\ \Delta^N x(t), & \nu = N. \end{cases}$$

Definition 3. [15] Let $x : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given, and $N - 1 < \nu \leq N$. Then the ν-th order Caputo difference of x is defined by

$${}^C\Delta_a^{\nu} x(t) = \begin{cases} \dfrac{1}{\Gamma(N-\nu)} \displaystyle\sum_{s=a}^{t-(N-\nu)} (t - \sigma(s))^{(N-\nu-1)} \Delta^N x(s), & N - 1 < \nu < N, \\ \Delta^N x(t), & \nu = N. \end{cases}$$

Lemma 1. *[1] Assume that $\nu > 0$ and $x : \mathbb{N}_a \to \mathbb{R}$, then,*

$$\Delta_{a+N-\nu}^{-\nu}\ {}^C\Delta_a^{\nu} x(t) = x(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{(k)}}{k!} \Delta^k x(a),\ t \in \mathbb{N}_{a+1},$$

where $N - 1 < \nu \leq N$.

Lemma 2. *[15] Let $a \in \mathbb{R}$ and $\mu > 0$ be given, then,*

$$\Delta(t-a)^{(\mu)} = \mu(t-a)^{(\mu-1)}.$$

Furthermore, for $\nu > 0$, the fractional difference and sum of the discrete power law function hold

$$\Delta_{a+\mu}^{-\nu}(t-a)^{(\mu)} = \mu^{(-\nu)}(t-a)^{(\mu+\nu)},\ t \in \mathbb{N}_{a+\mu+\nu},$$
$$\Delta_{a+\mu}^{\nu}(t-a)^{(\mu)} = \mu^{(\nu)}(t-a)^{(\mu-\nu)},\ t \in \mathbb{N}_{a+\mu+N-\nu}.$$

Particularly, the fractional sum of a constant C holds

$$\Delta_{a+1-\nu}^{-\nu} C = \frac{(t - \sigma(a) + \nu)^{(\nu)}}{\Gamma(\nu+1)} C,\ t \in \mathbb{N}_{a+1}.$$

3 Delay Mittag-Leffler functions

Let $a \in \mathbb{R}$ and $\nu > 0$. The delay discrete–time Mittag-Leffler function is defined in [1]

$$e_\nu(\lambda, (t-\sigma(a))^{(\nu)}) := \sum_{k=0}^{\infty} \frac{\lambda^k (t-a+k\nu-k)^{(k\nu)}}{\Gamma(k\nu+1)}, \ 0<\nu\leqslant 1, \ t\in\mathbb{N}_{a+1}. \quad (3.1)$$

We note that the function is a piece-wise one as

$$e_\nu(\lambda, (t-\sigma(a))^{(\nu)}) = \begin{cases} 1+\lambda, & t=a+1, \\ 1+\lambda(1+\nu)^{(\nu)}/\Gamma(\nu+1)+\lambda^2, & t=a+2, \\ \cdots & \\ \displaystyle\sum_{k=0}^{n} \frac{\lambda^k(n+k\nu-k)^{(k\nu)}}{\Gamma(k\nu+1)}, & t=a+n. \end{cases}$$

It is given in the form of a finite summation and there is no need to discuss the convergence. That is, λ is arbitrary. We give the asymptotic and non-negativity conditions as the following.

Theorem 1. *(Asymptoticity) [2] The delay discrete–time Mittag-Leffler function (3.1) is asymptotically stable*

$$\lim_{t\to\infty} e_\nu(\lambda, (t-\sigma(a))^{(\nu)}) = 0,$$

if $-2^\nu < \lambda < 0$.

Theorem 2. *(Non-negativity) The solution of the initial value problem*

$$\begin{cases} {}^C\Delta_a^\nu x(t) = \lambda x(t+\nu-1), \ 0<\nu\leqslant 1, \ t\in\mathbb{N}_{a+1-\nu}, \\ x(a)=1, \end{cases} \quad (3.2)$$

is non–negative if $\lambda > -\nu$.

Proof. We use the relationship of the Riemann-Liouville and Caputo difference:

$${}^C\Delta_a^\nu x(t) = \Delta_a^\nu (x(t)-x(a)).$$

So, Equation (3.2) can be rewritten as a fractional difference equation of R-L type:

$$\begin{cases} \Delta_a^\nu x(t) = x(a)(t-a)^{(-\nu)}/\Gamma(1-\nu) + \lambda x(t+\nu-1), \ t\in\mathbb{N}_{a+1-\nu}, \\ x(a)=1, \end{cases}$$

which leads to the numerical scheme:

$$\frac{1}{\Gamma(-\nu)} \sum_{j=0}^{n} \frac{\Gamma(n-j-\nu)}{\Gamma(n-j+1)} x(a+j) = x(a)\frac{\Gamma(n+1-\nu)}{\Gamma(1-\nu)\Gamma(n+1)} + \lambda x(a+n-1).$$

As a result, we collect

$$x(a+n) = x(a)\frac{\Gamma(n+1-\nu)}{\Gamma(1-\nu)\Gamma(n+1)} + (\lambda+\nu)x(a+n-1) - \frac{1}{\Gamma(-\nu)}\sum_{j=0}^{n-2}\frac{\Gamma(n-j-\nu)}{\Gamma(n-j+1)}x(a+j).$$

Since $0<\nu<1$ and $\Gamma(-\nu)<0$, the solution $x(t)$ is non-negative if $\lambda+\nu>0$. The proof is completed. □

4 Discrete Gronwall's inequality of delay type

Lemma 3. *Let $a\in\mathbb{R}$ and $\nu>0$. Suppose the following iteration equation holds*

$$\begin{cases} F_0(t) = \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1), \\ F_{k+1}(t) = \Delta_{a+1-\nu}^{-\nu} F_k(t+\nu-1),\ t\in\mathbb{N}_{a+1},\ k=0,1,2,\dots, \end{cases} \tag{4.1}$$

then,

$$F_k(t) = \sum_{s=a+1-\nu}^{t-\nu}\frac{(t+k(\nu-1)-\sigma(s))^{(k\nu+\nu-1)}}{\Gamma(k\nu+\nu)}h(s+\nu-1).$$

Proof. From (4.1), it can be seen that

$$F_0(t) = \sum_{s=a+1-\nu}^{t-\nu}\frac{(t-\sigma(s))^{(\nu-1)}}{\Gamma(\nu)}h(s+\nu-1),\ t\in\mathbb{N}_{a+1}.$$

Suppose that

$$F_k(t) = \sum_{s=a+1-\nu}^{t-\nu}\frac{(t+k(\nu-1)-\sigma(s))^{(k\nu+\nu-1)}}{\Gamma(k\nu+\nu)}h(s+\nu-1),\ t\in\mathbb{N}_{a+1}$$

holds, then for $k+1$

$$F_{k+1}(t) = \Delta_{a+1-\nu}^{-\nu}F_k(t+\nu-1) = \frac{1}{\Gamma(k\nu+\nu)\Gamma(\nu)} \times \sum_{r=a+1-\nu}^{t-\nu}(t-\sigma(r))^{(\nu-1)}\sum_{s=a+1-\nu}^{r-1}(r+(k+1)(\nu-1)-\sigma(s))^{(k\nu+\nu-1)}h(s+\nu-1).$$

By interchanging the order of summation, we obtain

$$F_{k+1}(t) = \frac{1}{\Gamma(k\nu+\nu)\Gamma(\nu)}\sum_{s=a+1-\nu}^{t-\nu-1}\sum_{r=s+1}^{t-\nu}(t-\sigma(r))^{(\nu-1)} \times (r+(k+1)(\nu-1)-\sigma(s))^{(k\nu+\nu-1)}h(s+\nu-1).$$

By Definition 1, we give

$$\frac{1}{\Gamma(k\nu+\nu)\Gamma(\nu)}\sum_{r=s+1}^{t-\nu}(t-\sigma(r))^{(\nu-1)}(r+(k+1)(\nu-1)-\sigma(s))^{(k\nu+\nu-1)}$$
$$\times h(s+\nu-1)=\Delta^{-\nu}_{b+(k+1)\nu-1-k}\frac{(t-b)^{(k\nu+\nu-1)}}{\Gamma(k\nu+\nu)}h(s+\nu-1)$$
$$=\Delta^{-\nu}_{b+(k+1)\nu-1}\frac{(t-b)^{(k\nu+\nu-1)}}{\Gamma(k\nu+\nu)}h(s+\nu-1),\ b=s-(k+1)\nu+2+k.$$

According to Lemma 2,

$$\Delta^{-\nu}_{b+(k+1)\nu-1}\frac{(t-b)^{(k\nu+\nu-1)}}{\Gamma(k\nu+\nu)}h(s+\nu-1)=\frac{(t-b)^{(k\nu+2\nu-1)}}{\Gamma(k\nu+2\nu)}h(s+\nu-1)$$
$$=\frac{(t+(k+1)(\nu-1)-\sigma(s))^{(k\nu+2\nu-1)}}{\Gamma(k\nu+2\nu)}h(s+\nu-1),\ t\in\mathbb{N}_{a+1}.$$

Therefore,

$$F_{k+1}(t)=\sum_{s=a+1-\nu}^{t-\nu-1}\frac{(t+(k+1)(\nu-1)-\sigma(s))^{(k\nu+2\nu-1)}}{\Gamma(k\nu+2\nu)}h(s+\nu-1)$$
$$=\sum_{s=a+1-\nu}^{t-\nu}\frac{(t+(k+1)(\nu-1)-\sigma(s))^{(k\nu+2\nu-1)}}{\Gamma(k\nu+2\nu)}h(s+\nu-1),\ t\in\mathbb{N}_{a+1},$$

which completes the proof. □

Theorem 3. *Let K be an arbitrary constant and λ be a non-negative one. If $x(t)$ satisfies*

$$x(t)\leqslant K+\lambda\Delta^{-\nu}_{a+1-\nu}x(t+\nu-1),\ t\in\mathbb{N}_{a+1},$$

then it is bounded by

$$x(t)\leqslant Ke_\nu(\lambda,(t-\sigma(a))^{(\nu)}),\ t\in\mathbb{N}_{a+1}.$$

Proof. Let $u(t)=K+\lambda\Delta^{-\nu}_{a+1-\nu}x(t+\nu-1),\ t\in\mathbb{N}_{a+1}$. Then, $x(t)\leqslant u(t)$ and $u(a)=K$. The following formula holds for $\lambda>0$

$${}^C\Delta^\nu_a u(t)=\lambda x(t+\nu-1)\leqslant\lambda u(t+\nu-1).$$

We construct the following nonhomogeneous equation:

$$\begin{cases}{}^C\Delta^\nu_a u(t)=\lambda u(t+\nu-1)-h(t+\nu-1),\ t\in\mathbb{N}_{a+1-\nu},\\ u(a)=K,\end{cases}$$

where $h(t+\nu-1)\geqslant 0$. Using Lemma 1, we get

$$u(t)=K+\Delta^{-\nu}_{a+1-\nu}(\lambda u(t+\nu-1)-h(t+\nu-1)),\ t\in\mathbb{N}_{a+1}.$$

By Picard's method, we get the successive iteration as

$$u_{m+1}(t) = u_0(t) + \lambda \Delta_{a+1-\nu}^{-\nu} u_m(t+\nu-1),\ m = 0,1,2,\ldots,$$

where $u_0(t) = K - \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1)$.

For $m = 0$, we get

$$\begin{aligned} u_1(t) =& u_0(t) + \lambda \Delta_{a+1-\nu}^{-\nu} u_0(t+\nu-1) \\ =& K - \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1) + \lambda \Delta_{a+1-\nu}^{-\nu}\Big(K - F_0(t+\nu-1)\Big) \\ =& K + K\frac{\lambda(t-a+\nu-1)^{(\nu)}}{\Gamma(\nu+1)} - F_0(t) - \lambda F_1(t), \end{aligned}$$

and for $m = 1$,

$$\begin{aligned} &u_2(t) = u_0(t) + \lambda \Delta_{a+1-\nu}^{-\nu} u_1(t+\nu-1) = K - \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1) \\ &+ \lambda \Delta_{a+1-\nu}^{-\nu}\Big(K + K\frac{\lambda(t-a+2\nu-2)^{(\nu)}}{\Gamma(\nu+1)} - F_0(t+\nu-1) - \lambda F_1(t+\nu-1)\Big) \\ &= K + K\frac{\lambda(t-a+\nu-1)^{(\nu)}}{\Gamma(\nu+1)} + K\frac{\lambda^2(t-a+2\nu-2)^{(2\nu)}}{\Gamma(2\nu+1)} - F_0(t) - \lambda F_1(t) \\ &- \lambda^2 F_2(t). \end{aligned}$$

More generally, we have

$$u_m(t) = K\left(1 + \sum_{k=1}^{m} \frac{\lambda^k (t-a+k\nu-k)^{(k\nu)}}{\Gamma(k\nu+1)}\right) - \sum_{k=0}^{m} \lambda^k F_k(t),$$

$$\sum_{k=0}^{m} \lambda^k F_k(t) = \sum_{k=0}^{m} \sum_{s=a+1-\nu}^{t-\nu} \frac{\lambda^k (t-\sigma(s)+k\nu-k)^{(k\nu+\nu-1)}}{\Gamma(k\nu+\nu)} h(s+\nu-1).$$

Let $m \to \infty$, then,

$$\begin{aligned} u(t) =& K\sum_{k=0}^{\infty} \frac{\lambda^k (t-a+k\nu-k)^{(k\nu)}}{\Gamma(k\nu+1)} \\ &- \sum_{s=a+1-\nu}^{t-\nu} \sum_{k=0}^{\infty} \frac{\lambda^k (t-\sigma(s)+k\nu-k)^{(k\nu+\nu-1)}}{\Gamma(k\nu+\nu)} h(s+\nu-1). \end{aligned}$$

Therefore, we obtain

$$x(t) \le K e_\nu(\lambda, (t-\sigma(a))^{(\nu)}),\ t \in \mathbb{N}_{a+1}.$$

□

Theorem 4. *Let $g : \mathbb{N}_a \to \mathbb{R}$ be a non-negative and non-decreasing function. K is an arbitrary constant. If $x(t)$ satisfies*

$$x(t) \leqslant K + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1) x(t+\nu-1),$$

then, the following inequality holds

$$x(t) \leqslant K e_\nu(g(t-1), (t-\sigma(a))^{(\nu)}),\ t \in \mathbb{N}_{a+1}.$$

Proof. Assume $u(t) = K + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)x(t+\nu-1),\ t \in \mathbb{N}_{a+1}$. Then, $x(t) \leqslant u(t)$ and $u(a) = K$. The following formula holds

$$^{C}\Delta_a^{\nu} u(t) = g(t+\nu-1)x(t+\nu-1) \leqslant g(t+\nu-1)u(t+\nu-1).$$

We use the following nonhomogeneous equation:

$$\begin{cases} ^{C}\Delta_a^{\nu} u(t) = g(t+\nu-1)u(t+\nu-1) - h(t+\nu-1),\ t \in \mathbb{N}_{a+1-\nu}, \\ u(a) = K, \end{cases}$$

where $h(t+\nu-1) \geqslant 0$.
By Picard's method, we get the successive iteration as

$$u_{m+1}(t) = u_0(t) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)u_m(t+\nu-1),\ m = 0, 1, 2, \dots,$$

where $u_0(t) = K - \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1)$.

Assume

$$\begin{cases} G_0(t) = K, \\ G_{k+1}(t) = \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)G_k(t+\nu-1),\ k = 0, 1, 2, \dots, \end{cases}$$

and

$$\begin{cases} H_0(t) = \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1), \\ H_{k+1}(t) = \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)H_k(t+\nu-1),\ k = 0, 1, 2, \dots,\ t \in \mathbb{N}_{a+1}. \end{cases}$$

For $m = 0$, we get

$$\begin{aligned} u_1(t) =& u_0(t) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)u_0(t+\nu-1) \\ =& K - \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)\Big(K - H_0(t+\nu-1)\Big) \\ =& G_0(t) + G_1(t) - H_0(t) - H_1(t). \end{aligned}$$

It can be obtained from the non–negativity of g

$$u_1(t) \leqslant G_0(t) + G_1(t),\ t \in \mathbb{N}_{a+1}.$$

g is a non-decreasing function such that

$$u_1(t) \leqslant K + K\frac{g(t-1)(t-a+\nu-1)^{(\nu)}}{\Gamma(\nu+1)},\ t \in \mathbb{N}_{a+1}.$$

For $m = 1$, we get

$$\begin{aligned} u_2(t) =& u_0(t) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)u_1(t+\nu-1) \\ =& K - \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1) \\ & \times \Big(K + KG_0(t+\nu-1) - H_0(t+\nu-1) - H_1(t+\nu-1)\Big) \\ =& G_0(t) + G_1(t) + G_2(t) - H_0(t) - H_1(t) - H_2(t). \end{aligned}$$

Since each $H_i(t)$ is non-negative, we give

$$u_2(t) \leqslant G_0(t) + G_1(t) + G_2(t),$$
$$u_2(t) \leqslant K + K\frac{g(t-1)(t-a+\nu-1)^{(\nu)}}{\Gamma(\nu+1)} + K\frac{g^2(t-1)(t-a+2\nu-2)^{(2\nu)}}{\Gamma(2\nu+1)}.$$

More generally, we have

$$u_m(t) \leqslant \sum_{k=0}^{m} G_k(t) - \sum_{k=0}^{m} H_k(t),$$
$$u_m(t) \leqslant K\sum_{k=0}^{m}\frac{g^k(t-1)(t-a+k\nu-k)^{(k\nu)}}{\Gamma(k\nu+1)},\ t\in\mathbb{N}_{a+1}.$$

Let $m\to\infty$

$$u(t) \leqslant K\sum_{k=0}^{\infty}\frac{g^k(t-1)(t-a+k\nu-k)^{(k\nu)}}{\Gamma(k\nu+1)}.$$

Therefore,

$$x(t) \le u(t) \le Ke_\nu(g(t-1),(t-\sigma(a))^{(\nu)}),\ t\in\mathbb{N}_{a+1}.$$

□

In [12], Ferreira gave a Gronwall's inequality for the Riemann-Liouville fractional difference equation. We give a more general one by Picard's method.

Theorem 5. *Let $g:\mathbb{N}_a\to\mathbb{R}$ be a non-decreasing and non-negative function. Let $q(t) = f(t)g(t)$ and $q:\mathbb{N}_a\to\mathbb{R}$ is a non-decreasing function. If $x(t)$ satisfies*

$$x(t) \leqslant f(t-1) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)x(t+\nu-1),$$

then $x(t)$ is bounded by

$$x(t) \leqslant f(t-1)e_\nu(g(t-1),(t-\sigma(a))^{(\nu)}),\ t\in\mathbb{N}_{a+1}.$$

Proof. Assume $u(t) = f(t-1) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)x(t+\nu-1),\ t\in\mathbb{N}_{a+1}$. Then $x(t)\leqslant u(t)$ and $u(a) = f(a-1)$. The following formula holds

$${}^C\Delta_a^\nu u(t) \leqslant {}^C\Delta_a^\nu f(t-1) + g(t+\nu-1)u(t+\nu-1).$$

We construct the nonhomogeneous equation:

$$\begin{cases} {}^C\Delta_a^\nu u(t) = {}^C\Delta_a^\nu f(t-1) + g(t+\nu-1)u(t+\nu-1) - h(t+\nu-1),\ t\in\mathbb{N}_{a+1-\nu}, \\ u(a) = K, \end{cases}$$

where $h(t+\nu-1)\geqslant 0$. By Picard's method, we get the successive iteration as

$$u_{m+1}(t) = u_0(t) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)u_m(t+\nu-1),\ m=0,1,\ldots,$$

where $u_0(t) = f(t-1) - \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1)$. Suppose that

$$\begin{cases} M_0(t) = f(t-1), \\ M_{k+1}(t) = \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1) M_k(t+\nu-1), \ k = 0, 1, \dots, \end{cases}$$

and

$$\begin{cases} H_0(t) = \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1), \\ H_{k+1}(t) = \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1) H_k(t+\nu-1), \ k = 0, 1, \dots, \ t \in \mathbb{N}_{a+1}. \end{cases}$$

For $m = 0$, we get

$$\begin{aligned} u_1(t) =& u_0(t) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1) u_0(t+\nu-1) \\ =& f(t-1) - \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1)\Big(f(t+\nu-2) \\ & - H_0(t+\nu-1)\Big) = M_0(t) + M_1(t) - H_0(t) - H_1(t). \end{aligned}$$

It can be obtained from the non–negativity of g

$$u_1(t) \leqslant M_0(t) + M_1(t).$$

Since $q(t) = f(t)g(t)$ is non-decreasing on $\mathbb{N}_a$, there is

$$u_1(t) \leqslant f(t-1) + f(t-1) \frac{g(t-1)(t-a+\nu-1)^{(\nu)}}{\Gamma(\nu+1)}.$$

For $m = 1$, we get

$$\begin{aligned} u_2(t) &= u_0(t) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1) u_1(t+\nu-1) \\ &= f(t-1) - \Delta_{a+1-\nu}^{-\nu} h(t+\nu-1) + \Delta_{a+1-\nu}^{-\nu} g(t+\nu-1) \\ &\times \Big(M_0(t+\nu-1) + M_1(t+\nu-1) - H_0(t+\nu-1) - H_1(t+\nu-1)\Big) \\ &= M_0(t) + M_1(t) + M_2(t) - H_0(t) - H_1(t) - H_2(t). \end{aligned}$$

It can be obtained from the non–negativity of function g

$$u_2(t) \leqslant M_0(t) + M_1(t) + M_2(t).$$

Because $q(t) = f(t)g(t)$ is non-decreasing on $\mathbb{N}_a$, there is

$$\begin{aligned} u_2(t) \leqslant & f(t-1) + f(t-1) \frac{g(t-1)(t-a+\nu-1)^{(\nu)}}{\Gamma(\nu+1)} \\ & + f(t-1) \frac{g^2(t-1)(t-a+2\nu-2)^{(2\nu)}}{\Gamma(2\nu+1)}. \end{aligned}$$

More generally, we have

$$u_m(t) \leqslant \sum_{k=0}^{m} M_k(t) - \sum_{k=0}^{m} H_k(t)$$

and

$$u_m(t) \leqslant f(t-1)\sum_{k=0}^{m}\frac{g^k(t-1)(t-a+k\nu-k)^{(k\nu)}}{\Gamma(k\nu+1)}.$$

Let $m \to \infty$

$$u(t) \leqslant f(t-1)\sum_{k=0}^{\infty}\frac{g^k(t-1)(t-a+k\nu-k)^{(k\nu)}}{\Gamma(k\nu+1)},$$

$$x(t) \leq f(t-1)e_\nu(g(t-1),(t-\sigma(a))^{(\nu)}),\ t\in\mathbb{N}_{a+1}.$$

□

Remark 1. We note that f can be a non-increasing function. Theorem 5 can be used in state estimation of the Riemann-Liouville fractional difference equation:

$$\Delta_a^\nu x(t) = \phi(x(t+\nu-1), t+\nu-1),\ 0<\nu\leq 1,\ t\in\mathbb{N}_{a+1-\nu},$$

where ϕ is well defined.

Suppose $x_n = x(a+n)$, $f_n = f(a+n)$ and $g_n = g(a+n)$. We can rewrite it as the following one directly.

Theorem 6. *Suppose $g:\mathbb{N}_a\to\mathbb{R}$ be a non-decreasing and non-negative function. Let $q(t)=f(t)g(t)$ and $q:\mathbb{N}_a\to\mathbb{R}$ is a non-decreasing function. The general inequality is given as*

$$x_{n+1} \leqslant f_n + \frac{1}{\Gamma(\nu)}\sum_{j=0}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}g_j x_j,$$

which yields

$$x_{n+1} \leqslant f_n e_\nu(g_n, n^{(\nu)}),\ n\geq 0.$$

Remark 2. Let $g:\mathbb{N}_a\to\mathbb{R}$ be a non-decreasing and non-negative function. Let $q(t)=f(t)g(t)$ and $q:\mathbb{N}_a\to\mathbb{R}$ is a non-decreasing function. By use of the h–fractional sum [21], a fractional sum inequality is given as

$$x_{n+1} \leqslant f_n + \frac{h^\nu}{\Gamma(\nu)}\sum_{j=0}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}g_j x_j,$$

which leads to

$$x_{n+1} \leqslant f_n e_\nu(h^\nu g_n, (hn)^{(\nu)}),\ n\geq 0.$$

Here the h-fractional Gronwall's inequality is different from the one in [20] where the conditions $g(t)\leq M$ and $h^\nu M<1$ are needed (see Theorem 3.1 of [20], pp. 820).

5 Ulam stability

In this section, we consider the following initial value problem of the fractional difference equation

$$\begin{cases} {}^{C}\Delta_a^{\nu} x(t) = \varphi(x(t+\nu-1), t+\nu-1), 0<\nu\leq 1,\ t\in\mathbb{N}_{a+1-\nu}, \\ x(a)=c, \end{cases} \tag{5.1}$$

where x and $\varphi(x(\cdot),\cdot):\mathbb{N}_a\to\mathbb{R}$. $\varphi(x,t)$ satisfies the Lipschitz condition

$$|\varphi(y,t)-\varphi(x,t)|\leqslant L|y(t)-x(t)|,\ L>0,\ t\in\mathbb{N}_{a+1}$$

and $\varphi(x,t)$ satisfies $\varphi(0,0)=0$.
The solution of (5.1) satisfies the fractional sum equation:

$$x(t)=c+\frac{1}{\Gamma(\nu)}\sum_{s=a+1-\nu}^{t-\nu}(t-\sigma(s))^{(\nu-1)}\varphi(x(s+\nu-1),s+\nu-1),\ t\in\mathbb{N}_{a+1}.$$

We need to introduce the following fixed point theorem to prove the uniqueness of the solution of Equation (5.1).

Lemma 4. *[7, 9] (Banach fixed point theorem) Let $X=(X,d)$ be a nonempty complete metric space, $T:X\to X$ is a compressed mapping on X, then T has exactly one fixed point.*

Let B as the set of all $x=\{x(t)\}_{t\in\mathbb{N}_a}$ with the norm $\|x\|=\sup_{t\in\mathbb{N}_a}|x(t)|$. Then, B is a Banach space. Suppose $M>0$, we define the set $S=\{x(t)\mid t\in\mathbb{N}_a,\ \|x\|\leqslant M\}$.

Theorem 7. *Equation (5.1) has a unique solution on S if there exists $0<r<1$ such that*

$$L\big((t-\sigma(a)+\nu)^{(\nu)}\big)/\Gamma(\nu+1)<r,\ t\in\mathbb{N}_{a+1}.$$

Proof. Define the operator

$$(Tx)(t)=c+\frac{1}{\Gamma(\nu)}\sum_{s=a+1-\nu}^{t-\nu}(t-\sigma(s))^{(\nu-1)}\varphi(x(s+\nu-1),s+\nu-1),\ t\in\mathbb{N}_{a+1}.$$

Obviously, $x(t)$ is a solution of (5.1) if it is a fixed point of the operator T.

By Lemma 4, we can know that to prove that problem (5.1) has a unique solution, that is, to prove that T is a contractive mapping.

First, we prove that T maps S in S. Assume that $|c|\leq(1-r)M$. Then,

$$\begin{aligned}|(Tx)(t)|\leq&|c|+\Big|\frac{1}{\Gamma(\nu)}\sum_{s=a+1-\nu}^{t-\nu}(t-\sigma(s))^{(\nu-1)}\varphi(x(s+\nu-1),s+\nu-1)\Big|\\ \leq&|c|+L\frac{1}{\Gamma(\nu)}\sum_{s=a+1-\nu}^{t-\nu}(t-\sigma(s))^{(\nu-1)}|x(s+\nu-1)|\\ \leq&|c|+ML\frac{(t-\sigma(a)+\nu)^{(\nu)}}{\Gamma(\nu+1)}\leq M.\end{aligned}$$

So, we have $\|(Tx)(t)\| \le M$, which implies that T maps S in S.

And also because

$$|(Ty)(t) - (Tx)(t)| = |\frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} (t-\sigma(s))^{(\nu-1)} (\varphi(y(s+\nu-1), s+\nu-1) - \varphi(x(s+\nu-1), s+\nu-1))|$$
$$\le L \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} (t-\sigma(s))^{(\nu-1)} |y(s+\nu-1) - x(s+\nu-1)|$$
$$\le L \frac{\Gamma(t-a+\nu)}{\Gamma(\nu+1)\Gamma(t-a)} |y(s+\nu-1) - x(s+\nu-1)| < r|y(t) - x(t)|,$$

we have

$$|(Ty)(t) - (Tx)(t)| < r|y(t) - x(t)|.$$

Since $0 < r < 1$, this shows that T is a contraction mapping. By Banach fixed point theorem, the $T : S \to S$ has a unique fixed point $x(t)$ which is a also unique solution of the initial value problem (5.1). □

Similar as the definitions of Ulam stability for fractional differential equation [25], we introduce the one of delay fractional difference equations. We consider the fractional difference equation with delay (5.1) and the following inequality

$$|{}^C\Delta_a^\nu y(t) - \varphi(y(t+\nu-1), t+\nu-1)| \le \varepsilon, \ t \in \mathbb{N}_{a+1-\nu}. \tag{5.2}$$

Definition 4. [10] For $w : \mathbb{N}_a \to \mathbb{R}$, Equation (5.1) is Hyers-Ulam-Rassias stable if for each $\varepsilon > 0$ and each solution $y : \mathbb{N}_a \to \mathbb{R}$ of the inequality (5.2), there exists a solution $x \in S$ of Equation (5.1) with

$$|y(t) - x(t)| \le w(t)\varepsilon, \ t \in \mathbb{N}_{a+1}.$$

Theorem 8. *(Ulam stability theorm) Suppose the Lipschitz condition of φ holds. Let $y : \mathbb{N}_a \to \mathbb{R}$ be a solution of the inequality (5.2) and $x \in S$ be a solution of the Cauchy problem*

$$\begin{cases} {}^C\Delta_a^\nu x(t) = \varphi(x(t+\nu-1), t+\nu-1), \quad 0 < \nu \le 1, \ t \in \mathbb{N}_{a+1-\nu}, \\ x(a) = y(a). \end{cases} \tag{5.3}$$

Then, Equation (5.1) is Hyers-Ulam-Rassias stable.

Proof. The solution of the Cauchy problem (5.3) is given by

$$x(t) = y(a) + \Delta_{a+1-\nu}^{-\nu} \varphi(x(t+\nu-1), t+\nu-1), \ t \in \mathbb{N}_{a+1}.$$

By the inequality (5.2), we have

$$|y(t) - y(a) - \Delta_{a+1-\nu}^{-\nu} \varphi(y(t+\nu-1), t+\nu-1)|$$
$$\le \Delta_{a+1-\nu}^{-\nu} \varepsilon = \frac{\Gamma(t-a+\nu)}{\Gamma(\nu+1)\Gamma(t-a)} \varepsilon.$$

It follows that

$$\begin{aligned}|y(t)-x(t)| &= |y(t)-y(a)-\Delta_{a+1-\nu}^{-\nu}\varphi(x(t+\nu-1),t+\nu-1)| \\ &\le |y(t)-y(a)-\Delta_{a+1-\nu}^{-\nu}\varphi(y(t+\nu-1),t+\nu-1)| \\ &+ |\Delta_{a+1-\nu}^{-\nu}(\varphi(y(t+\nu-1),t+\nu-1)-\varphi(x(t+\nu-1),t+\nu-1))| \\ &\le \frac{\Gamma(t-a+\nu)}{\Gamma(\nu+1)\Gamma(t-a)}\varepsilon + L\Delta_{a+1-\nu}^{-\nu}|y(t+\nu-1)-x(t+\nu-1)|.\end{aligned}$$

Using Theorem 5, we have

$$|y(t)-x(t)| \le \frac{\Gamma(t-a+\nu)}{\Gamma(\nu+1)\Gamma(t-a)} e_\nu(L,(t-\sigma(a))^{(\nu)})\varepsilon,\ t\in\mathbb{N}_{a+1}.$$

Thus, Equation (5.1) is Hyers-Ulam-Rassias stable. The proof is completed. □

In the proof of the above theorem, we use

$$w(t) = \frac{\Gamma(t-a+\nu)}{\Gamma(\nu+1)\Gamma(t-a)} e_\nu(L,(t-\sigma(a))^{(\nu)}),\ t\in\mathbb{N}_{a+1}.$$

If $t=a+n, n\ge 1$, then,

$$w(a+n) = \frac{\Gamma(n+\nu)}{\Gamma(\nu+1)\Gamma(n)} \sum_{k=0}^{n} \frac{L^k(n+k\nu-k)^{(k\nu)}}{\Gamma(k\nu+1)}.$$

We take the following numerical example to support the theoretical analysis.

Example 1. Consider the following fractional sine map's initial value problem

$$\begin{cases} {}^C\Delta_a^\nu x(t) = \mu\sin x(t+\nu-1),\ 0<\nu\le 1,\ t\in\mathbb{N}_{a+1-\nu}, \\ x(a)=0.5. \end{cases} \tag{5.4}$$

Let the fractional order $\nu=0.8$, $\varepsilon=0.05$ and $t=0,\dots,10$. We have the contractive mapping constant $r=0.8$, the Lipshitz constant $\mu=0.1$ and $M=4$. If $y(t)$ is an approximate solution of (5.4) with the numerical error ε

$${}^C\Delta_a^\nu y(t) = \mu\sin y(t+\nu-1)+\varepsilon,\ t\in\mathbb{N}_{a+1-\nu},$$

according to Definition 4 and Theorem 8, $|y(t)-x(t)|$ can be estimated as

$$|y(t)-x(t)| \le w(t)\varepsilon,\ t\in\mathbb{N}_{a+1}.$$

Figure 1 gives the exact solution $x(t)$ and the approximate solution $y(t)$ for $t\in\{0,\dots,10\}$, respectively. Figure 2 shows that the relationship $|y(t)-x(t)|\le w(t)\varepsilon$ holds and Equation (5.4) is Hyers-Ulam-Rassias stable.

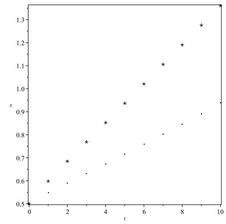

Figure 1. Exact solution (the point) and approximate solution (the asterisk).

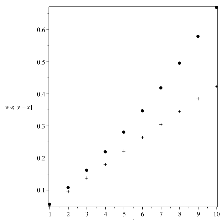

Figure 2. $w(t)\varepsilon$ (the point) and $|y(t)-x(t)|$ (the cross).

6 Conclusions

This paper presents a discrete Gronwall's inequality and Ulam stability of delay fractional difference equations. Because fractional recurrent neural networks [16] can be described by this fractional difference equation, this paper contributes some basics of Ulam stability of fractional discrete neural networks.

For example, we can obtain one approximate solution $y(t)$ of Equation (5.1) with a known residual error ε. According to the Ulam stability condition, we can obtain the error estimation between the exact solution $x(t)$ as:

$$|y(t)-x(t)| \leq w(t)\varepsilon.$$

We will consider these possible applications in future work.

Acknowledgements

This work is financially supported by the National Natural Science Foundation of China (NSFC) (Grant No. 62076141) and Sichuan Youth Science and Technology Foundation (Grant No. 2022JDJQ0046).

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