Mathematical Modelling and Analysis Publisher: Taylor&Francis and VGTU
Volume 17 Number 1, February 2012, 100–112 http://www.tandfonline.com/TMMA
http://dx.doi.org/10.3846/13926292.2012.645251 Print ISSN: 1392-6292
Online ISSN: 1648-3510

On Integral Representation of the Translation Operator

Paulius Miškinis

Faculty of Fundamental Sciences, Vilnius Gediminas Technical University
Saulėtekio al. 11, LT-10223 Vilnius, Lithuania
E-mail: paulius.miskinis@vgtu.lt

Received March 13, 2011; revised September 26, 2011; published online February 1, 2012

Abstract. The formulation in the explicit form of quantum expression of the one-dimensional translation operator as well as Hermitian operator of momentum and its eigenfunctions are presented. The interrelation between the momentum and the wave number has been generalized for the processes with a non-integer dimensionality α. The proof of the fractional representation of the translation operator is considered. Some aspects of the translations in graduate spaces and their integral representation, as well as applications in physics are discussed. The integral representation of the translation operator is proposed.

Keywords: traslation operator, quantum mechanics, fractional calculus.

AMS Subject Classification: 31A10, 34K37, 81Q35.

1 Introduction

The translation operator $\hat{T}_a$ is an operator in which

$$\hat{T}_a f(x) = f(x+a), \tag{1.1}$$

where a is a parameter of the translation. For the smooth functions $f(x)$, the corresponding generator is the exponential map of an ordinary derivative [2]:

$$\hat{T}_a = \mathsf{e}^{a\frac{d}{dx}}. \tag{1.2}$$

In the case of non-smooth functions when the ordinary derivative does not exist, we have to generalize the corresponding representation (1.2), substituting the ordinary derivative by the fractional one and turning to the generalized exponential function. The idea of this generalization was first mentioned in the monograph [19].

Axiomatization of the nation of translation operator (1.1) has resulted in important generalization concerning spaces of functions on groups, i.e. the concept of generalized translation operator, or hypergroup [1]. In terms of generalized translation operator, it is possible to formulate such important mathematical concepts as convolution, group algebra, positively defined function, almost periodic function, etc. In particular, the theory of generalized translation operator has essential application in abstract harmonic analysis [11]. Worth noting that in the theory of differential equations, the concept of translation operator along the trajectories of differential equation, which has no direct relation to functional analysis, is also denoted as translation operator [7]. Similarly as the concept of generalized solution, it is usually not expressed by an explicit formula; however, numerous properties of translation operator may be derived directly from the properties of differential equation. In the theory of differential equation, the term of translation operator is not commonly accepted; also the term "mapping for the time t_0 to t" and some others are used in this sense. On the other hand, the development of computation technologies has stimulated the application of the term "translation operator" in the Hilbert space as sequences $l^p[\Omega]$, which actually has no direct relation to the concept of translation operator (1.1) employed in the present work.

The physical basis for the existence of quantum mechanics comprises a series of phenomena described by the mathematical theory of the Wienerian processes.

The Lévy stochastic process is a natural generalization of the Brownian motion or the Wiener stochastic process [4]. The foundation for this generalization is the theory of stable probability distributions developed by Lévy [12]. The most fundamental property of the Lévy distributions is the stability with respect to addition, in accordance with the generalized central limit theorem. Thus, from the probability theory point of view, the stable probability law is a generalization of the well-known Gaussian law. The Lévy processes are characterized by the Lévy index α, which takes values $0 < \alpha \leqslant 2$. At $\alpha = 2$ we have the Gaussian process or the process of the Brownian motion. Lévy process is widely used to model a variety of processes such as anomalous diffusion [14], turbulence [5], chaotic dynamics [22], plasma physics [24], financial dynamics [15], biology and physiology [21] (for recent references see e.g. [3, 18, 23]) .

$$\begin{tikzcd}
\mathrm{QM} \arrow[r, "\alpha"] & \mathrm{FQM} \\
\mathrm{WP} \arrow[u, "h"] \arrow[r, "\alpha"'] & \mathrm{LP} \arrow[u, "h"']
\end{tikzcd}$$

Figure 1. Schematic representation of interrelations of Wienerian processes (WP), Lévy processes (LP), quantum mechanics (QM) and fractional quantum mechanics (FQM).

The constantly increasing number of experimental facts in various fields of knowledge related to classical non-Wienerian processes [4, 12, 14] evokes a natural desire to "close" the commutative diagram shown in Fig. 1 and, at least formally, to consider the possible existence of a quantum analogue of a more narrow class of phenomena related to Lévy processes, the so-called fractional quantum mechanics (FQM) [8, 10, 13].

The present note offers a brief discussion of one of the crucial issues related to FQM, which is the one-dimensional operator of momentum. Like in usual quantum mechanics (QM), one-dimensional problems are a kind of excess idealization. Nevertheless, they may be used for elucidating the fundamental features of FQM. One-dimensional problems arise while considering the three-dimensional evolutionary equation in which the interaction potential depends on a single coordinate. This fact allows, with the aid of a corresponding factorization, to move to a simpler one-dimensional evolutionary equation.

The purpose of this paper is formulation in the explicit form quantum expression of the one-dimensional operator of momentum for the fractional probability processes. The proof of the fractional representation of the translation operator and the corresponding integral representation is considered. Some aspects of the translations in graduate spaces and their integral representation, as well as applications in physics are discussed.

2 Fractional Quantum Operator of Momentum

The classical definition of momentum in QM follows from the invariance of the Hamiltonian of the quantum system $\hat{H}$ with respect to the infinitesimal displacements δx. Under such transformation, the wave function $\psi(x)$ turns into the function

$$\psi(x+\delta x) = \psi(x) + \delta x\, \partial_x \psi = (1 + \delta x\, \partial_x)\, \psi(x),$$

here ∂_x is the differentiation operator over the space variable x [6, 9].

However, it may turn out that $\partial_x \psi$ does not exist, but there exists the so-called fractional derivative $\partial_x^\alpha \psi$ in which the order of the derivative α may be both an integer and a fractional number. For the function determined on the whole real axis $\mathbb{R}$, the right and left derivatives of the order α are derived as

$$\partial_\pm^\alpha \psi(x) = \frac{\{\alpha\}}{\Gamma(1-\{\alpha\})} \int_0^\infty \frac{\psi^{([\alpha])}(x) - \psi^{([\alpha])}(x \mp \xi)}{\xi^{1+\{\alpha\}}}\, d\xi,$$

where $[\alpha]$ and $\{\alpha\}$ are the integer and the fractional parts of the parameter α (for details, see Appendix). For the bilateral derivatives to exist, it is sufficient that $\psi(x) \in C^{[\alpha]}(\Omega)$, where $C^{[\alpha]}(\Omega)$ is a set of continuously differentiated functions of the order $[\alpha]$ determined on the domain Ω [18].

Another peculiarity related to the operator of momentum is the expansion of the wave-function $\psi(x)$ into a Taylor series by fractional powers [23]

$$\psi(x) = \sum_{n=0}^{[\alpha]} c_n^{(\alpha)}\, (x - x_0)^{\alpha+n} + R_n(x), \tag{2.1}$$

where $c_n^{(\alpha)}$ are numerical coefficients and $R_n(x)$ is the residual term, which provides a better approach to the initial function. In all such cases, determination of the quantum operator of momentum should be specified.

It is reasonable to suppose that the momentum operator should be proportional to the fractional derivative:

$$\hat{p} = C\partial_+^\alpha \psi(x) \tag{2.2}$$

here C is a certain coefficient of proportionality. For $\alpha \to 0$, we must obtain a usual quantum operator of momentum, $\hat{p} = -i\hbar\partial_x$. Thus, in FMQ we always deal with two kinds of limit transitions: (1) $\hbar \to 0$, when we shift to classical mechanics, and (2) $\alpha \to 0$, when we turn to usual QM (see Fig. 1).

The kind of the coefficient C in the expression for the momentum (2.2) is best defined if on the whole real axis we consider a flat wave of the form

$$\psi(x,t) = A\, e^{i\kappa\, x - i\frac{E}{\hbar}t}, \quad \kappa = \left(\frac{p}{\hbar l_0^{\alpha-1}}\right)^{\frac{1}{\alpha}}, \tag{2.3}$$

there $\hbar$ is the Planck constant and l_0 is a certain peculiar scale of the length of the nonlocal process under consideration.

Let us impose a requirement for the momentum operator (2.2) to obey the eigenvalue equation $\hat{p}\psi = p\psi$. Applying the property of the fractional derivatives, $\partial_+^\alpha e^{\kappa x} = \kappa^\alpha e^{\kappa x}$ $(\mathrm{Re}\,\kappa > 0)$ (see Appendix), we obtain that

$$C = (-i)^\alpha \hbar l_0^{\alpha-1}. \tag{2.4}$$

For the eigenvalues observed in QM to be real, the corresponding operators should be Hermitian. It is easy to see that the quantum operator of momentum (2.2) with the constant C from (2.4) is non-Hermitian. In order to obtain a Hermitian operator of momentum, to the type (2.2) operator we will add a Hermite-conjugated operator $\hat{p}^+$; then, the momentum operator determined in this way

$$\hat{p} = \frac{\hbar l_0^{\alpha-1}}{2}\left[(-i\partial_+)^\alpha + (i\tilde{\partial}_-)^\alpha\right], \tag{2.5}$$

here $\sim$ is the symbol of transposition, will be clearly Hermitian. Indeed, the momentum operator (2.5) $\hat{p} = (\hat{p}_+ + \hat{p}_-)/2$ will be Hermitian because of the idempotency of the operation of Hermitian conjugation $((\hat{p}^+)^+ = \hat{p})$ and the structure of the operator itself $(\hat{p}_- = (\hat{p}_+)^+)$. On the other hand, employing the rule of fractional integration by parts (see Appendix),

$$p^* = \int_{-\infty}^{+\infty} \varphi^*\, \hat{p}^+ \psi\, dx = \frac{1}{2}\int_{-\infty}^{+\infty} \varphi^*(\hat{p}_- + \hat{p}_+)\psi\, dx = p,$$

we directly see that the momentum operator is Hermitian for the different functions of state φ and ψ.

Thus, we obtain that the operator (2.5) is Hermitian and its eigenvalues on the whole real axis are flat waves of the (2.3) type. Like in the classical case, the eigenvalues of the momentum operator do not belong to the class $L^2(\mathbb{R})$. Therefore, they don't describe the physically realizable states of the quantum particle. These eigenfunctions should be regarded as the basic functions, which comprise the complete system of functions.

3 Wave Function Normalization

To determine the constant normalization A in the expression for the flat wave (2.3), we will take that $\int \psi_0^* \psi_p \, dx = \delta(p)$. This is a particular case of the conventional condition $\int \psi_{p'} \psi_p dx = \delta(p' - p)$ for $p' \equiv 0$. Using the property of the δ-function, we shall obtain that

$$A = \sqrt{\frac{|\kappa|}{2\pi |p|}}, \quad \kappa \equiv \left(\frac{p}{\hbar l_0^{\alpha-1}} \right)^{\frac{1}{\alpha}}. \tag{3.1}$$

Let us specify the peculiarities of such normalization. Firstly, generally speaking, the amplitude is a complex magnitude; secondly, it depends on the eigenvalue of the momentum p . Only when $\alpha \to 1$, as the case should be, $A \to 1/\sqrt{2\pi\hbar}$.

Inasmuch as the physical sense applies not to the amplitude itself but to $|A|^2$, the complex nature of A does not contradict unitarity. However, because of the complex nature of the amplitude we may get an impression that we deal with a damping wave; however, actually there is no damping, because $A \neq A(x,t)$. Besides, the same conclusion results from analysis of the dispersion expression of the corresponding Hamiltonian. The dependence $A = A(p)$ is not a matter of principle and may be avoided by a suitable choice of the normalization condition. For instance, under condition $\int \psi_{\kappa'}^* \psi_\kappa \, dx = \delta(\kappa' - \kappa)$, the dependence $A(p)$ is absent.

Another important circumstance should be noted as regards the type of the momentum operator. Transition to momentum representation is not a Fourier transformation. Momentum representation should be understood in the sense of f-representation:

$$\psi(x) = \int a_f \psi_f(x) \, df, \quad \int \psi_{f'}^* \psi_f \, dx = \delta(f' - f).$$

4 Fractional Translation Operator

Lastly, let us derive the formula to express, through the momentum operator $\hat{p}$, the parallel translation operator in space to any finite (not only infinitesimal) distance. From the definition of such an operator it follows:

$$i\hat{T}_{a+}^\alpha \psi(x) = \psi(x - a), \quad \hat{T}_{b-}^\alpha \psi(x) = \psi(x + b).$$

In this case, a and b denote the values of finite displacements but not the coordinate ends of the interval.

Expanding the function $\psi(x - a)$ in the neighbourhood of the point x into a Taylor series by fractional powers as in (2.1) and employing the expression for the "right-hand" and "left-hand" parts of the momentum operator,

$$\hat{p}_+^\alpha = -i\hbar l_0^{\alpha-1} \partial_+^\alpha, \qquad \hat{p}_-^\alpha = i\hbar l_0^{\alpha-1} \tilde{\partial}_-^\alpha,$$

we obtain that

$$\hat{T}_{a+}^\alpha = \sum_{k=0}^{\infty} \frac{(ia)^{\alpha+k-1}}{(\hbar l_0^{\alpha-1})^{(\alpha+k-1)/\alpha}} \frac{\hat{p}_+^{(\alpha+k-1)}}{\Gamma(\alpha + k)} \equiv E_{1-\alpha}^{\kappa_a \hat{p}_+^\alpha},$$

where E_μ^z is the generalized exponential function (see Appendix). These are exactly the finite displacement operators we have been searching for.

For $\alpha \to 1$, for $\hat{T}_{a+}^{\alpha}$ we obtain that

$$\hat{T}_{a+}^{\alpha} \to \sum_{k=0}^{\infty} \left(\frac{ia}{\hbar}\right)^k \frac{\hat{p}_+^k}{k!} \equiv e^{i\frac{a}{\hbar}\hat{p}_+}, \tag{4.1}$$

The expression $\hat{T}_{b-}^{\alpha}$ could be obtain by substituting in equation (4.1) $a \to -b$ and $\hat{p}_+^{\alpha} \to \hat{p}_-^{\alpha}$.

5 Integral Representation of the Translation Operator

Thus, in both the classical and quantum cases, the translation operator may be represented as an exponent (see (1.2) and (4.1)). An exponent, and in a broader sense also a power series, may be represented as an integral.

Lemma 1. *The residue of the Euler γ-function is*

$$Res_n\, \Gamma(-z) = (-1)^{n+1}/n!.$$

Proof. Let us use the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, which is valid not only for real, but also for complex $z \in \mathbb{C}$. From it, let us express $\Gamma(z)$ and substitute $-z$ for z:

$$\Gamma(-z) = -\frac{\pi}{\sin\pi z}\frac{1}{\Gamma(1+z)}.$$

Let us calculate the residue at the point $z = n$:

$$Res_n\, \Gamma(-z) = \lim_{z\to n} \Gamma(-z)(z-n) = -\frac{1}{\cos\pi n}\frac{1}{\Gamma(1+n)} = \frac{(-1)^{(n+1)}}{n!}. \tag{5.1}$$

The lemma is proved. □

The above lemma allows finding an integral representation of the power series for which Theorem 1 is valid.

Theorem 1.

$$\sum_{n=0}^{\infty} f(n)\frac{z^n}{n!} = \frac{1}{2\pi i}\oint_\gamma f(t)\Gamma(-t)(-z)^t\, dt. \tag{5.2}$$

Proof. In a complex plane, let us consider the integral $\oint_\gamma f(t)\Gamma(-t)(-z)^t\, dt$, in which the contour γ does not cover the singularities of the complex-valued function $f(z)$. Suppose that it is possible to close the contour γ on the right side, i.e. the contour is circumvent in the negative direction (Fig. 2). From the theorem of the residue of the complex function $f(z)$ [17] and the above lemma it follows:

$$\oint\limits_\gamma f(t)\Gamma(-t)(-z)^t\, dt = -2\pi i \sum_{n=0}^{\infty} f(n)\frac{(-1)^{(n+1)}}{n!}(-z)^n = 2\pi i \sum_{n=0}^{\infty} f(n)\frac{z^n}{n!}.$$

Dividing both parts of this expression by $2\pi i$ gives the statement of the theorem. □

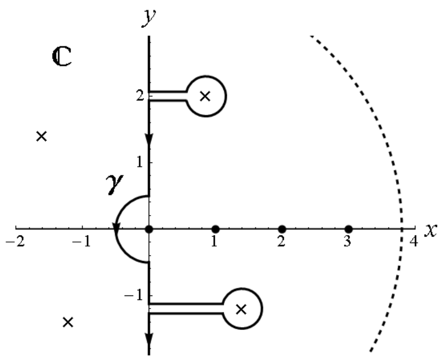

Figure 2. The singular points of the function $\Gamma(-z)$ on the complex plane $\mathbb{C}$ are marked with a dot "•". The possible singular points of the function $f(z)$ are marked with "×". The contour γ is supposed to be closed on the right side.

Corollary 1. The classical right-translation operator has the following integral representation

$$\hat{T}_{a+} = e^{-a\frac{d}{dx}} = \frac{1}{2\pi i}\oint_{\gamma} \Gamma(-t)a^t \partial^t_{x+}\, dt. \tag{5.3}$$

Corollary 2.

$$\hat{T}^{\alpha}_{a+} = E^{\kappa_a \hat{p}^{\alpha}_{+}}_{1-\alpha} = \frac{1}{2\pi i}\oint_{\gamma} \Gamma(-t)a^{\alpha+t} \partial^{\alpha+t}_{x+}\, dt. \tag{5.4}$$

Corollary 3.

$$\frac{d}{dx} = \log\Big[\frac{1}{2\pi i}\oint_{\gamma} \Gamma(-t)\partial^t_{x+}\, dt\Big]. \tag{5.5}$$

Corollary 4.

$$\partial^{\alpha}_{x+} = \log\Big[\frac{1}{2\pi i}\oint_{\gamma} \Gamma(-t)\partial^{\alpha+t-1}_{x+}\, dt\Big]. \tag{5.6}$$

Proof. By applying (5.2) to the exponential representation (1.2) we obtain the statement of Corollary 1. Note here, the order of the fractional derivative in (5.3) and below is a complex value. The effect of the fractional derivative ∂^{α}_{x+} on both sides of expression (5.3) gives Corollary 2. The logarithmic function from both parts of expression (5.3) for $a = 1$ gives (5.5). Finally, substituting $f(x) \to \partial^{\alpha-1}_{x+} f(x)$ in expression (5.5) for df/dx results in (5.6). Thus, Corollaries 1–4 are proved. □

6 The Simplest Nonlocal Superalgebra

The simplest nonlocal supergroup generators have the following representation

$$\{Q^{\alpha}_{\sigma}, Q^{\alpha}_{\rho}\} = 2\,(\gamma^{\mu}C)_{\sigma\rho}\, P^{\alpha}_{\mu}, \quad \left[Q^{\alpha}_{\sigma}, P^{\alpha}_{\mu}\right] = 0,$$

where Q^{α}_{σ} is the generator of supertranslations, $\sigma, \rho = 1, 2$, γ^{μ} are the Dirac γ-matrices, $C_{\sigma\rho}$ is the matrix of charge conjugation and P^{α}_{μ} is the four-dimensional operator of momenta with fractional order α.

We are interested in the explicit form of these generators, similarly as the generator

$$P^{\alpha}_{\mu} = \frac{1}{2}\left[(-i\partial_{\mu+})^{\alpha} + (i\tilde{\partial}_{\mu-})^{\alpha}\right] \equiv \partial^{\alpha}_{\mu},$$

is a translation generator in the space $X^{\alpha} \ni x$ with fractional dimension α ($\hbar = l_0^{\alpha-1} = 1$). Now we have to expand the concept of space-time so as it could include the supersymmetric partner of coordinate x. Let us denote the superspace produced by the couple x_{μ}, θ_{σ}. The supersymmetry generator would be defined as follows:

$$Q^{\alpha}_{\sigma} = \frac{\partial}{\partial\bar{\theta}^{\sigma}} + (\gamma^{\mu}\theta)_{\sigma}\, \partial^{\alpha}_{\mu},$$

where θ is the Grassmann number. We choose exactly this representation, because, as could be expected, the anticommutator between two such generators, gives the space translation

$$\{Q^{\alpha}_{\sigma}, Q^{\alpha}_{\rho}\} = 2\,(\gamma^{\mu}C)_{\sigma\rho}\, \partial^{\alpha}_{\mu}.$$

Note that $\bar{\varepsilon}Q$ performs superspace transformations:

$$x^{\mu} \to x^{\mu} - \bar{\varepsilon}\gamma^{\mu}\theta, \quad \theta^{\,\sigma} \to \theta^{\,\sigma} + \varepsilon^{\sigma}.$$

Also, note that it is possible to create the operator

$$D^{\alpha}_{\sigma} = \frac{\partial}{\partial\bar{\theta}^{\sigma}} + \left(\gamma^{\mu}\partial^{\alpha}_{\mu}\theta\right)_{\sigma},$$

which anticommutes with the supersymmetry generator $\left\{Q^{\alpha}_{\sigma}, D^{\alpha}_{\rho}\right\} = 0$. It is very important, because we may impose constraints on supersymmetry representations without destroying this symmetry. This allows deriving reduced representations from nonreduced ones.

7 Conclusions

Note that the classical restriction on the smoothness of the wave function $\psi(x) \in C^2([a, b])$ does not hold here. The restriction on $\psi(x)$ follows from the continuity equation; however, in the case of fractional dimension we can show that the condition of continuity is changed, and the limitation on $\psi(x)$ is reduced to $\psi(x) \in C^{[\alpha]}([a, b])$.

Another note pertains to the structure of the momentum operator. It seems highly significant that the momentum operator consists of two parts—the right and left displacements. In classical fractional mechanics, it is quite possible to limit ourselves to one of these two components, $\hat{p}_+$ or $\hat{p}_-$. In the quantum case, it is impossible to take such a limit, because the full operator of momentum is a Hermitian.

The limit transition $\hbar \to 0$ for $\{\alpha\} \neq 0$ means transition to classical fractional mechanics. However, the form of the momentum operator undergoes no qualitative change: $\hat{p} = \frac{1}{2}(\hat{p}_+ + \hat{p}_-)$, i.e. it consists of two parts, each being proportional to its one-sided derivative. For linear evolutionary equations of classical (not quantum) fractional mechanics this type of structure of the momentum operator may be simplified if $\hat{p} = \hat{p}_+$ or $\hat{p} = \hat{p}_-$. However, here additional considerations are necessary. For the nonlinear fractional evolutionary processes it is impossible in principle, because $\hat{p}_+ = \hat{p}_-$ is the condition of smoothness.

From the definition of κ there follows an interrelation between the momentum and the wave number $p = \hbar l_0^{\alpha-1}\kappa^\alpha$, for $\alpha \to 1$, $\kappa \to k$ this expression turns into $p = \hbar k$.

The appearance of the characteristic length scale of l_0 and the power dependence of the quantum particle momentum on the wave number directly indicate the fractional character of quantum mechanics.

Thus we have the Hermitian quantum operator of momentum (2.5) with the eigenfunctions (2.3). This allows us to construct the quadratic form of the Hamiltonian $\hat{H} \propto \hat{p}^2$ instead of the nonlinear form $\hat{H} \propto D_\alpha |p|^\alpha$ [8], and the Hermitian Hamiltonian instead of non-Hermitian proposed in [10], and the Unitarian Hamiltonian instead of non-unitarian proposed in [13].

The following aspect deserves special attention. According to the theory of dynamical systems, there is a strict border-line between the local and nonlocal evolutionary systems. Therefore, the integral presentation of the translation operator and the resulting consequences seem very intriguing. It appears that the classical local operator of the derivative may be presented in an absolutely nonlocal form (see (5.5)). This implies that local dynamical systems are a particular case of nonlocal systems which are much less studied.

Appendix A

The right and left fractional derivatives are defined in the form

$$\partial_\pm^\alpha f(x) = \frac{\{\alpha\}}{\Gamma(1-\{\alpha\})} \int_0^{+\infty} \frac{f^{[\alpha]}(x) - f^{[\alpha]}(x \mp \xi)}{\xi^{1+\{\alpha\}}}\, d\xi, \tag{A.1}$$

where $\Gamma(z)$ is the Euler γ-function, $\alpha = [\alpha] + \{\alpha\}$ are the integer and the fractional parts of the real number $\alpha \in \mathbb{R}$. These are the so-called Marchaud derivatives [16], which on the whole real axis are more natural than, e.g., the Riemann–Liouville derivatives. For instance, for the functions determined on $\mathbb{R} : f(x) \in L^p$, where $1 \leqslant p < 1/\alpha$,

$$\partial_\pm^\alpha I_\pm^\alpha f(x) = f(x), \tag{A.2}$$

here $I_\pm^\alpha f(x)$ is a fractional integral of the order α, whereas for the Riemann–Liouville derivatives the property (A.2) holds only if $\alpha = 1$. The Marchaud derivatives, on sufficiently "good" functions, coincide with the Riemann–Liouville derivatives, however, in contrast to the latter, they allow even an increase of the functions of the order below α on the infinity. The differences between the Riemann–Liouville and Marchaud functions, related to the behaviour on the infinity, are absent in the case of a finite interval [20].

Remind here that the one-parametrical family of linear limited operators $\{T_\alpha\}$, $\alpha \geqslant 0$ in the Banach space X comprises a semi-group, if

$$T_\alpha T_\beta = T_{\alpha+\beta}, \quad \alpha \geqslant 0,\ \beta \geqslant 0, \qquad T_0\varphi = \varphi, \quad \forall\varphi \in X.$$

A semigroup of operators is called strongly continuous if

$$\lim_{\alpha\to\alpha_0} \|T_\alpha\varphi - T_{\alpha_0}\varphi\| = 0, \quad 0 \leqslant \alpha_0 < \infty,\ \forall\varphi \in X.$$

From the semigroup character of the family $\{\partial^\alpha_\pm\}$ (A.1) it follows that if a semigroup is strongly continuous when $\alpha = 0$, it is inevitably strongly continuous for all $\alpha \geqslant 0$.

From the definition of the fractional operators it follows that fractional integration operators comprise in $L^p(a,b)$, $p \geqslant 1$ a semigroup, which is continuous in the uniform topology for all $\alpha > 0$. $L^p(a,b)$, as usual, denote a set of functions $|f|^p$ in the p power, which are measurable according to Lebesgue.

The form of the approximation of the operator I^α_{a+} to unity when $\alpha \to 0$ is conditioned by the form of the generating operator $\mathcal{L}(x) \equiv \lim_{\alpha\to+0} \|I^\alpha_{a+}\varphi - \varphi\|/\alpha$. The expression for the operator is calculated from the definition using the L'Hôpital rule. The calculation gives the following expression for the form of $I^\alpha_{a+}\varphi$ approximation:

$$\mathcal{L}(x) = \frac{d}{dx}\int_a^x \ln(x-t)\varphi(x)\,dt - \Gamma'(1)\varphi(x),$$

for almost all x. It is convenient to make use of unified designation also for the other integrals and derivatives, considering that

$$\partial^\alpha_+ f = I^{-\alpha}_+ f = \left(I^\alpha_+\right) f, \quad \alpha > 0.$$

The semigroup character of the fractional derivatives is

$$\partial^\alpha_\pm \partial^\beta_\pm f(x) = \partial^\beta_\pm \partial^\alpha_\pm f(x) = \partial^{\alpha+\beta}_\pm f(x).$$

The fractional integration by parts is

$$\int_a^b \varphi(x)\partial^\alpha_+\psi(x)\,dx = \int_a^b \psi(x)\partial^\alpha_-\varphi(x)\,dx,$$

$$\begin{aligned} &\textit{Example 1a:} \quad \partial^\alpha_+ e^{\lambda x+\mu} = \lambda^\alpha e^{\lambda x+\mu} \quad (\operatorname{Re}\lambda > 0),\\ &\textit{Example 1b:} \quad \partial^\alpha_- e^{\lambda x+\mu} = (-\lambda)^\alpha e^{\lambda x+\mu} \quad (\operatorname{Re}\lambda < 0). \end{aligned}$$

Some special functions are a very convenient tool for applications, e.g., the Mittag-Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0,\ \beta > 0.$$

It is obvious that $E_{1,1}(z) = e^z$.

Another example is the fractional exponential function:

$$E_\mu^z = \sum_{k=0}^{\infty} \frac{z^{k-\mu}}{\Gamma(k-\mu+1)} = e^{-\mu} E_{1,1-\mu}(z) = \partial_+^{(\mu)} e^z, \quad \alpha > 0,\ \beta > 0.$$

Note here that the E_μ^z function for the fractional translation operator $\hat{T}^\alpha$ plays the same role as a usual exponent function e^z for the translation operator of the integer order. The list could be supplemented by Wright function, Fox H-functions, *etc.*, [23].

Appendix B

Traditional symmetries in physics are generated by objects that transform under the tensor representations of the Poincaré group and internal symmetries. Supersymmetries, on the other hand, are generated by objects that transform under the spinor representations. According to the spin-statistics theorem, bosonic fields commute while fermionic fields anticommute. Combining the two kinds of fields into a single algebra requires the introduction of a $\mathbb{Z}_2$-grading under which the bosons are the even elements and the fermions are the odd elements. Such algebra is called a Lie superalgebra. Thus, superalgebra is a $\mathbb{Z}_2$-graded algebra over a commutative ring or field with a decomposition into "even" and "odd" pieces and a multiplication operator that respects the grading.

Formal definition. Let K be a fixed commutative ring. In most applications, K is a field such as $\mathbb{R}$ or $\mathbb{C}$. A superalgebra over K is a K-module $\mathcal{A}$ with a direct sum decomposition $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, together with a bilinear multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that

$$\mathcal{A}_i \mathcal{A}_j = \mathcal{A}_{i+j},$$

where the subscripts are read modulo 2.

A superring, or $\mathbb{Z}_2$-graded ring, is a superalgebra over the ring of integers $\mathbb{Z}$.

The elements of $\mathcal{A}_i$ are said to be homogeneous. The parity of a homogeneous element x, denoted by $|x|$, is 0 or 1 according to whether it is in $\mathcal{A}_0$ or $\mathcal{A}_1$. Elements of parity 0 are said to be even and those of parity 1 to be odd. If x and y are both homogeneous then so is the product xy and $|xy| = |x| + |y|$.

An associative superalgebra is one whose multiplication is associative and a unital superalgebra is one with a multiplicative identity element. The identity element in a unital superalgebra is necessarily even. A commutative superalgebra is one which satisfies a graded version of commutativity. Specifically, $\mathcal{A}$ is commutative if

$$yx = (-1)^{|x|\,|y|} xy,$$

for all homogeneous elements x and y of $\mathcal{A}$.

Some examples.

- any exterior algebra over K is a superalgebra. The exterior algebra is the standard example of a supercommutative algebra.

- Clifford algebras are superalgebras. They are generally noncommutative.
- Lie superalgebras are a graded analog of Lie_algebras. Lie superalgebras re nonunital and nonassociative; however, one may construct the analog of a universal enveloping algebra of a Lie superalgebra which is a unite, associative superalgebra. There are representations of a Lie superalgebra that are analogous to representations of a Lie algebra. Each Lie algebra has an associated Lie group and a Lie superalgebra can sometimes be extended into representations of a Lie supergroup.

The prefix *super-* comes from the theory of supersymmetry in theoretical physics. Superalgebras and their representations, supermodules, provide an algebraic framework for formulating supersymmetry. The study of such objects is sometimes called super linear algebra. Superalgebras also play an important role in related field of supergeometry where they enter into the definitions of graded manifolds, supermanifolds and superschemes.

References

[1] J. Delsarte. Hypergroupes et opérateurs de permutation et de transmutation. *Colloq. Internat. CNRS*, **71**:29–45, 1956.

[2] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov and R.G. Burns. *Modern Geometry – Methods and Applications: Part I: The Geometry of Surfaces, Transformation Groups, and Fields (Grad. Texts in Math.) (Pt. 1)*. Springer, New York, 1991.

[3] A. Le Mehauté et al. (Eds.). *Fractional Differentiation and Its Applications*. Books on Demand, Norderstedt, 2005.

[4] M. Kac. *Probability and Related Topics in Physical Sciences*. Interscience Publishers, New York, 1959.

[5] J. Klafter, A. Blumen and M.F. Shlesinger. Stochastic pathway to anomalous diffusion. *Phys. Rev. A*, **35**(7):3081–3085, 1987. http://dx.doi.org/10.1103/PhysRevA.35.3081.

[6] H. Kleinert. *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*. World Scientific, Singapore, 1990.

[7] M.A. Krasnosel'sky. *A Shift Operator along the Trajectories of Differential Equations*. Nauka, Moscow, 1966. (in Russian)

[8] D. Kusnezov, A. Bulgac and G. Do Dang. Quantum Lévy processes and fractional kinetics. *Phys. Rev. Lett.*, **82**(6):1136–1139, 1999. http://dx.doi.org/10.1103/PhysRevLett.82.1136.

[9] L.D. Landau and E.M. Lifshitz. *Quantum Mechanics, Course of Theoretical Physics, vol. 3 (Nonrelativistic Theory)*. Pergamon, New York, 1965.

[10] N. Laskin. Fractional quantum mechanics. *Phys. Rev. E*, **62**(3):3135–3145, 2000. http://dx.doi.org/10.1103/PhysRevE.62.3135.

[11] B.M. Levitan. *Theory of Generalized Shift Operators*. Nauka, Moscow, 1973. (in Russian)

[12] P. Lévy. *The'orie de l'Addition des Variables Aléatoires*. Gauthier-Villars, Paris, 1937.

[13] E. Lutz. Fractional transport equations for Lévy stable processes. *Phys. Rev. Lett.*, **86**(11):2208–2211, 2001.
http://dx.doi.org/10.1103/PhysRevLett.86.2208.

[14] B.B. Mandelbrot and J.W. van Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Rev.*, **10**:422–437, 1968.
http://dx.doi.org/10.1137/1010093.

[15] R.N. Mantega and H.E. Stanley. Scaling behavior in the dynamics of an economic index. *Nature*, **376**(6535):46–48, 1995. http://dx.doi.org/10.1038/376046a0.

[16] A. Marchaud. Sur les dérivées et sur les differences des functions de variables réelles. *J. Math. Pures Appl.*, **6**(4):337–425, 1927.

[17] A. I. Markushevich. *Theory of Functions of a Complex Variable.* Prentice–Hall, Englewood Cliffs, NJ, 1965.

[18] R. Metzler and J. Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Phys. A*, **37**(31):R161–R208, 2004.
http://dx.doi.org/10.1088/0305-4470/37/31/R01.

[19] P. Miskinis. *Nonlinear and Nonlocal Integrable Models.* Technika, Vilnius, 2004. (in Lithuanian)

[20] S.G. Samko, A.A. Kilbas and O.I. Marichev. *Fractional Integrals and Derivatives. Theory and Applications.* Gordon and Breach, Amsterdam, 1993.

[21] B.J. West and W. Deering. Fractal physiology for physicists: Lévy statistics. *Phys. Rep.*, **246**(1–2):1–153, 1994.
http://dx.doi.org/10.1016/0370-1573(94)00055-7.

[22] G.M. Zaslavsky. Fractional kinetic equation for hamiltonian chaos. *Phys. D*, **76**(1–3):110–122, 1994.

[23] G.M. Zaslavsky. Chaos, fractional kinetics, and anomalous transport. *Phys. Rep.*, **371**(6):461–580, 2002. http://dx.doi.org/10.1016/S0370-1573(02)00331-9.

[24] G. Zimbardo, P. Veltri, G. Basile and S. Principato. Anomalous diffusion and Lévy random walk of magnetic field lines in three dimensional turbulence. *Phys. Plasmas*, **2**(7):2653–2663, 1995. http://dx.doi.org/10.1063/1.871453.