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# CONVERGENCE RATE OF RATIONAL SPLINE HISTOPOLATION

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**Abstract.** The convergence rate of histopolation on arbitrary nonuniform mesh with linear/linear rational splines of class $C^1$ is studied. Established convergence rate depends on Lipschitz smoothness class of the function to histopolate. Corresponding numerical examples are given.

**Key words:** histopolation, rational spline, convergence rate

## 1. Introduction

It is well known that interpolating cubic and quadratic splines have good convergence properties for properly smooth functions. On the other hand, geometrical properties like monotonicity and convexity need not be preserved. The same could be said about histopolation with these splines. A natural way to preserve geometrical properties is the use of rational splines because, e.g., linear/linear rational splines of class $C^1$ are always monotone by itself. We have studied the histopolation with them in [1, 2] and we outlined in [1] without details a way to prove the convergence indicating also the convergence rate in the case of uniform mesh. In this paper we will give complete proofs of convergence rate for arbitrary meshes and for several Lipschitz smoothness classes of functions to histopolate.

The convergence of interpolating splines is better studied than that of histopolating ones [3, 5]. Instead of histopolation with linear/linear rational splines the corresponding interpolating problem may be formulated in a standard way and then the derivative of its solution is a solution of initial histopolation problem. Such an interpolation problem could be solved (see [4]) using quadratic/linear rational splines of class $C^2$ which are convex by itself. But the derivative of a quadratic/linear rational function, in general, is not linear/linear rational function and this algorithm produces a different method compared to the direct solution of histopolation problem with linear/linear rational splines. Therefore, the study of convergence rate for histopolation with linear/linear rational splines needs independent research.

## 2. Linear/Linear Rational Spline Histopolation to Monotone Data

In this section we introduce the notation and review basic results about a monotonicity preserving method of histopolation studied in [1].

Consider a given mesh $a = x_0 < x_1 < \ldots < x_n = b$ with given numbers $z_i$, $i = 1, \ldots, n$, i.e. a given histogram. A linear/linear rational spline histopolant is a $C^1$ smooth function $S$ on $[a, b]$ of the form

$$S(x) = \frac{a_i + b_i\Big(x - x_{i-1}\Big)}{1 + d_i\Big(x - x_{i-1}\Big)}$$

with $1 + d_i(x - x_{i-1}) > 0$ for $x \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$, such that

$$\int\limits_{x_{i-1}}^{x_i} S(x)dx = z_i\Big(x_i - x_{i-1}\Big), \quad i = 1, \ldots, n. \tag{2.1}$$

In addition, we impose the boundary conditions

$$S'(x_0) = \alpha, \quad S'(x_n) = \beta \tag{2.2}$$

or

$$S(x_0) = \alpha, \quad S(x_n) = \beta \tag{2.3}$$

for given $\alpha$ and $\beta$.

In such a general situation, there are no two different linear/linear rational splines of class $C^1$ satisfying histopolation conditions (2.1) and boundary conditions (2.2) or (2.3) (see [1]).

On the other hand, a $C^1$ linear/linear rational spline is strictly increasing or strictly decreasing or constant on $[a, b]$. This implies that, for the existence of the solution, it is necessary that

$$z_1 < \ldots < z_n \quad \text{or} \quad z_1 > \ldots > z_n \quad \text{or} \quad z_1 = \ldots = z_n \tag{2.4}$$

and the boundary data have to be consistent with histogram data. We proved in [1] that, on the assumptions (2.4) with consistent boundary data, the solution of considered form exists.

The actual construction of histopolating spline could be implemented solving a nonlinear system of basic equations

$$m_i = \varphi_i(m) := \frac{\delta_i}{h_i\varphi\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big) + h_{i+1}\varphi\Big(\Big(\frac{m_{i+1}}{m_i}\Big)^{1/2}\Big)} \tag{2.5}$$

with two additional equations obtained from (2.2) or (2.3). Here we mean $m_i = S'(x_i)$, $i = 0, \dots, n$, $m = (m_0, \dots, m_n)$, $h_i = x_i - x_{i-1}$, $i = 1, \dots, n$, $\delta_i = z_{i+1} - z_i$ and $\varphi$ is the function

$$\varphi(x) = \begin{cases} \dfrac{x^2(\log x - 1) + x}{(x-1)^2} & \text{for } x > 0\,,\ x \neq 1\,, \\ 0.5 & \text{for } x = 1\,. \end{cases}$$

Furthermore, we stipulate that the right hand side of (2.5) defines the functions $\varphi_i$. The conditions (2.2) simply fix the values $m_0 = \alpha$ and $m_n = \beta$, but, e.g., the condition $S(x_0) = \alpha$ leads to the equation

$$m_0 = \varphi_0(m) := \frac{\delta_0}{h_1 \varphi\Big(\Big(\dfrac{m_1}{m_0}\Big)^{1/2}\Big)} \tag{2.6}$$

with $\delta_0 = z_1 - \alpha$. The equation (2.6) could be considered as special case of (2.5) where $i = 0$ and $h_0 = 0$, $z_0 = \alpha$.

Clearly, only the case $z_1 < \ldots < z_n$ needs the study and we restrict ourselves to that. Suppose we have a given (at least, differentiable) function $f$ on $[a, b]$. The numbers $z_i$ are calculated as

$$z_i = \frac{1}{h_i} \int\limits_{x_{i-1}}^{x_i} f(x)\, dx, \quad i = 1, \dots, n,$$

the conditions (2.2) are of the form $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$ and (2.3) are $S(x_0) = f(x_0)$, $S(x_n) = f(x_n)$. The strict increase of histogram heights, i.e. $z_i < z_{i+1}$ for all $i$, is guaranteed, e.g., by $f'(x) > 0$ for all $x \in [a, b]$.

We are interested in the convergence rate of $(S - f)$ in uniform norm on $[a, b]$ as $h = \max_{1 \le i \le n} h_i$ goes to zero.

## 3. Estimates of First Moments

We derive our convergence rate results basing on the estimates of $m_i$ which will be established in this section.

**Lemma 1.** *Suppose $f' \in \operatorname{Lip} \alpha$ for some $\alpha \in (0, 1]$ and $f'(x) > 0$ for all $x \in [a, b]$. Then*

$$m_i - f'(x_i) = \mathcal{O}(h^\alpha).$$

*Proof.* Take $K_i = [f'(x_i) - ch^\alpha, f'(x_i) + ch^\alpha]$ with a number $c > 0$ independent of $h$ and which will be specified later. Showing that $\varphi_i : \prod_{i=0}^n K_i \to K_i$ for all $i$, we may use Bohl-Brouwer fixed point theorem and the uniqueness of the solution of the system $m_i = \varphi_i(m)$, $i = 0, \dots, n$, to state that $m_i \in [f'(x_i) - ch^\alpha, f'(x_i) + ch^\alpha]$.

First, let us analyze the main case $i = 1, \dots, n - 1$. Using in integrals of

$$\delta_i = \frac{1}{h_{i+1}} \int\limits_{x_i}^{x_{i+1}} f(x)\,dx - \frac{1}{h_i} \int\limits_{x_{i-1}}^{x_i} f(x)\,dx$$

the Taylor expansion $f(x) = f(x_i) + f'(x_i)(x - x_i) + R$, where it holds $|R| \le (L/(1+\alpha))|x - x_i|^{1+\alpha}$ and $L$ is the Lipschitz constant of $f'$, we get

$$\delta_i = \frac{1}{2}\Big(h_i + h_{i+1}\Big) f'(x_i) \pm \frac{L}{(1+\alpha)(2+\alpha)}\Big(h_i^{1+\alpha} + h_{i+1}^{1+\alpha}\Big). \tag{3.1}$$

The compact writing $p = q \pm r$, as usual, denotes the two-sided inequality $q - r \le p \le q + r$.

Next, consider the expansion

$$\varphi\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big) = \varphi(1) + \varphi'(\xi_i)\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2} - 1\Big), \quad \xi_i \in \Big(1, \Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big). \tag{3.2}$$

Choose $m_i = f'(x_i) \pm ch^\alpha$, $i = 0, \dots, n$. Then we obtain

$$\frac{m_{i-1}}{m_i} - 1 = \frac{m_{i-1} - m_i}{m_i} = \pm \frac{2c + L}{f'(x_i) - ch^\alpha} h^\alpha. \tag{3.3}$$

Let us remark, in addition, that this yields $m_{i-1}/m_i \to 1$ as $h \to 0$. Using the Taylor expansion up to the second derivative for $(1+x)^{1/2}$ at 0, we obtain

$$\sqrt{x} - 1 = \sqrt{1 + (x-1)} - 1 = \frac{x-1}{2} - \frac{(x-1)^2}{8(1+\xi)^{3/2}}, \quad \xi \in (0, x-1).$$

This, applied in the case $x = m_{i-1}/m_i$ with the help of (3.3) leads to

$$\begin{aligned} \Big(\frac{m_{i-1}}{m_i}\Big)^{1/2} - 1 &= \frac{1}{2}\Big(\frac{m_{i-1}}{m_i} - 1\Big) + \mathcal{O}(h^{1+\alpha}) \\ &= \pm\Big(\frac{c + \frac{L}{2}}{f'(x_i) - ch^\alpha} h^\alpha + \mathcal{O}(h^{1+\alpha})\Big) = \pm\Big(\frac{c + \frac{L}{2}}{f'(x_i)} h^\alpha + \mathcal{O}(h^{2\alpha})\Big). \end{aligned} \tag{3.4}$$

We may conclude that, in (3.2) and then in (2.5), it holds

$$\varphi'(\xi_i) = \frac{1}{3} + \mathcal{O}(h^\alpha). \tag{3.5}$$

Analogous calculations could be done for the term $\varphi((m_{i+1}/m_i)^{1/2})$ in (2.5).

Taking in (2.5) into account (3.1) and (3.2), (3.4), (3.5) with their counterparts for $m_{i+1}$, we obtain

$$\varphi_i(m) = \frac{\frac{1}{2}\big(h_i + h_{i+1}\big) f'(x_i) \pm \frac{L}{(1+\alpha)(2+\alpha)}\big(h_i^{1+\alpha} + h_{i+1}^{1+\alpha}\big)}{\frac{1}{2}\big(h_i + h_{i+1}\big) \pm \big(h_i + h_{i+1}\big)\Big(\frac{1}{3} + \mathcal{O}(h^\alpha)\Big)\Big(\frac{c + \frac{L}{2}}{f'(x_i)} h^\alpha + \mathcal{O}(h^{2\alpha})\Big)}$$

$$= \frac{f'(x_i) \pm \frac{2L}{(1+\alpha)(2+\alpha)} h^\alpha}{1 \pm \left(\frac{1}{3} + \mathcal{O}(h^\alpha)\right)\left(\frac{2c+L}{f'(x_i)} h^\alpha + \mathcal{O}(h^{2\alpha})\right)}$$

$$= \left(f'(x_i) \pm \frac{2L}{(1+\alpha)(2+\alpha)} h^\alpha\right)\left(1 \pm \left(\frac{\frac{2}{3}c + \frac{L}{3}}{f'(x_i)} h^\alpha + \mathcal{O}(h^{2\alpha})\right)\right)$$

$$= f'(x_i) \pm \left(\left(\frac{2}{3}c + \left(\frac{1}{3} + \frac{2}{(1+\alpha)(2+\alpha)}\right)L\right)h^\alpha + Mh^{2\alpha}\right) \qquad (3.6)$$

with certain $M > 0$ depending, however, on $c$, $L$ and $f$. We have the inclusion $\varphi_i(m) \in K_i$ if

$$\frac{2}{3}c + \left(\frac{1}{3} + \frac{2}{(1+\alpha)(2+\alpha)}\right)L + Mh^\alpha \le c,$$

which, in turn, takes place for sufficiently large $c$ (e.g., in the case $\alpha = 1$, for $c > 2L$) and small $h$.

The boundary condition $S'(x_0) = f'(x_0)$ do not need any analysis and we deal briefly with $S(x_0) = f(x_0)$ leading to (2.6). Then it holds

$$\delta_0 = \frac{h_1}{2} f'(x_0) \pm \frac{L}{(1+\alpha)(2+\alpha)} h_1^{1+\alpha}$$

and with the help of the expansion

$$\varphi\left(\left(\frac{m_1}{m_0}\right)^{1/2}\right) = \varphi(1) + \varphi'(\xi_0)\left(\left(\frac{m_1}{m_0}\right)^{1/2} - 1\right), \quad \xi_0 \in \left(1, \left(\frac{m_1}{m_0}\right)^{1/2}\right),$$

we get for $\varphi_0$ the same final form of two-sided estimate (3.6). This completes the proof. ■

**Lemma 2.** *Suppose* $f'' \in \operatorname{Lip}\alpha$ *for some* $\alpha \in (0,1]$ *and* $f'(x) > 0$ *for all* $x \in [a,b]$. *Then*

$$m_i - f'(x_i) = \mathcal{O}(h^{1+\alpha}).$$

*Proof.* Let us write equations (2.5) in the form

$$F_i(m_{i-1}, m_i, m_{i+1}) = h_i m_i \varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) + h_{i+1} m_i \varphi\left(\left(\frac{m_{i+1}}{m_i}\right)^{1/2}\right) = \delta_i,$$
$$i = 1, \ldots, n-1, \qquad (3.7)$$

introducing at the same time functions $F_i$. By Taylor expansion we establish

$$\delta_i = \frac{h_i + h_{i+1}}{2} f'(x_i) + \frac{h_{i+1}^2 - h_i^2}{6} f''(x_i) + \mathcal{O}\left(h_i^{2+\alpha} + h_{i+1}^{2+\alpha}\right). \qquad (3.8)$$

At left hand side of (3.7) we use the Taylor expansion

$$F_i\left(m_{i-1}, m_i, m_{i+1}\right) = F_i\left(m_i, m_i, m_i\right) + F_i'\left(m_i, m_i, m_i\right)\overline{h}_i + \frac{F_i''(\xi_\lambda)}{2!}\overline{h}_i^2$$

with $\overline{h}_i = (m_{i-1} - m_i, 0, m_{i+1} - m_i)$, some $\lambda \in (0,1)$ and $\xi_\lambda = (m_i, m_i, m_i) + \lambda \overline{h}_i$. Here we have at once $F_i(m_i, m_i, m_i) = \frac{1}{2}(h_i + h_{i+1})m_i$. Concerning the term with $F_i'$ we calculate

$$\frac{\partial F_i}{\partial m_{i-1}} = \frac{h_i}{2}\varphi'\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big)\Big(\frac{m_{i-1}}{m_i}\Big)^{-1/2}$$

which gives $\frac{\partial F_i}{\partial m_{i-1}}(m_i, m_i, m_i) = \frac{1}{6}h_i$ and similarly we obtain the value $\frac{\partial F_i}{\partial m_{i+1}}(m_i, m_i, m_i) = \frac{1}{6}h_{i+1}$. In $F_i''$ we actually need only

$$\frac{\partial^2 F_i}{\partial m_{i-1}^2} = \frac{h_i}{4m_{i-1}}\Big(\varphi''\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big) - \varphi'\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big)\Big(\frac{m_{i-1}}{m_i}\Big)^{-1/2}\Big)$$

and similar derivative $\dfrac{\partial^2 F_i}{\partial m_{i+1}^2}$. Observe that, by Lemma 1, it holds $m_i \in [c_1, c_2]$ with $c_1, c_2 > 0$ for sufficiently small values of $h$. This gives

$$\lambda m_{i-1} + (1-\lambda)m_i, \quad \lambda m_{i+1} + (1-\lambda)m_i \in [c_1, c_2].$$

After standard calculations we can conclude that

$$\frac{F_i''(\xi_\lambda)}{2}\,\overline{h}_i^2 = h_i\alpha_i\big(m_{i-1} - m_i\big)^2 + h_{i+1}\beta_i\big(m_{i+1} - m_i\big)^2,$$

where $\alpha_i$ and $\beta_i$ are bounded. Thus, the left hand side of (3.7) reduces to

$$\frac{1}{6}h_i m_{i-1} + \frac{1}{3}\Big(h_i + h_{i+1}\Big)m_i + \frac{1}{6}h_{i+1}m_{i+1} + h_i\alpha_i\big(m_{i-1} - m_i\big)^2 + h_{i+1}\beta_i\big(m_{i+1} - m_i\big)^2. \quad (3.9)$$

In addition, using the formulae

$$f'(x_i) - h_i f''(x_i) = f'(x_{i-1}) + \mathcal{O}(h_i^{1+\alpha}),$$
$$f'(x_i) + h_{i+1} f''(x_i) = f'(x_{i+1}) + \mathcal{O}(h_{i+1}^{1+\alpha})$$

let us write (3.8) as

$$\delta_i = \frac{h_i}{6}f'(x_{i-1}) + \frac{h_i + h_{i+1}}{3}f'(x_i) + \frac{h_{i+1}}{6}f'(x_{i+1}) + \mathcal{O}\big(h_i^{2+\alpha} + h_{i+1}^{2+\alpha}\big). \quad (3.10)$$

Now (3.9) and (3.10) permit to transform (3.7) to the form

$$\lambda_i\big(m_{i-1} - f'(x_{i-1})\big) + 2\big(m_i - f'(x_i)\big) + \mu_i\big(m_{i+1} - f'(x_{i+1})\big) = -6\lambda_i\alpha_i\big(m_{i-1} - m_i\big)^2 - 6\mu_i\beta_i\big(m_{i+1} - m_i\big)^2 + \mathcal{O}\big(h^{1+\alpha}\big) \quad (3.11)$$

with $\lambda_i = \dfrac{h_i}{h_i + h_{i+1}}$ and $\mu_i = 1 - \lambda_i$.

In the case of boundary condition $S(x_0) = f(x_0)$ write (2.6) as

$$F_0(m_0, m_1) = h_1 m_0 \varphi\Big(\Big(\frac{m_1}{m_0}\Big)^{1/2}\Big) = \delta_0\,. \tag{3.12}$$

Here we use the expansions

$$\delta_0 = \frac{h_1}{2} f'(x_0) + \frac{h_1^2}{6} f''(x_0) + \mathcal{O}\big(h_1^{2+\alpha}\big) = \frac{h_1}{3} f'(x_0) + \frac{h_1}{6} f'(x_1) + \mathcal{O}\big(h_1^{2+\alpha}\big)$$

and

$$F_0(m_0, m_1) = F_0(m_0, m_0) + F_0'(m_0, m_0)\overline{h}_0 + \frac{F_0''(\xi_\lambda)}{2!}\,\overline{h}_0^2$$

with $\overline{h}_0 = (0, m_1 - m_0)$ and $\xi_\lambda = (m_0, m_0) + \lambda \overline{h}_0$. In the last formula we have

$$F_0(m_0, m_0) = \frac{h_1}{2} m_0, \quad \frac{\partial F_0}{\partial m_1}(m_0, m_0) = \frac{h_1}{6}, \quad \frac{F_0''(\xi_\lambda)}{2}\,\overline{h}_0^2 = h_1 \alpha_0 (m_1 - m_0)^2,$$

where $\alpha_0$ is bounded. The equation (3.12) takes the form

$$2\big(m_0 - f'(x_0)\big) + \big(m_1 - f'(x_1)\big) = -6\alpha_0 \big(m_1 - m_0\big)^2 + \mathcal{O}\big(h_1^{1+\alpha}\big). \tag{3.13}$$

Observe that the assumption $f'' \in \operatorname{Lip}\alpha$ guarantees $f' \in \operatorname{Lip} 1$ and then, by Lemma 1, $(m_i - m_{i-1}) = \mathcal{O}(h)$ or $(m_i - m_{i-1})^2 = \mathcal{O}(h^2)$ for all $i$.

Considering now the equations (3.11) and (3.13) with its analogue at $x_n$ as a linear system with respect to $m_i - f'(x_i)$, $i = 0, \dots, n$, we find out that there is the diagonal dominance in rows. However, the condition $S'(x_0) = f'(x_0)$ gives the trivial equation $m_0 - f'(x_0) = 0$ which preserves the property of diagonal dominance. This yields

$$m_i - f'(x_i) = \mathcal{O}(h^{1+\alpha}),$$

which completes the proof. ■

*Remark 1.* Instead of exact boundary conditions $S(x_0) = f(x_0)$ and $S'(x_0) = f'(x_0)$ it may be used their perturbed versions $S(x_0) = f(x_0) + \mathcal{O}(h_1^{1+\alpha})$ and $S'(x_0) = f'(x_0) + \mathcal{O}(h_1^{\alpha})$ in Lemma 1, as well $S(x_0) = f(x_0) + \mathcal{O}(h_1^{2+\alpha})$ and $S'(x_0) = f'(x_0) + \mathcal{O}(h_1^{1+\alpha})$ in Lemma 2.

## 4. Convergence Estimates

In this section we establish the convergence rate of uniform norm

$$\|S - f\|_\infty = \max_{a \le x \le b} |S(x) - f(x)|$$

for $S$ being the linear/linear rational spline histopolant to a function $f$ as was described in Section 2. In addition, the convergence rate of $\|S' - f'\|_\infty$ is obtained.

**Lemma 3.** *In the assumptions of Lemma 1 (respectively, of Lemma 2) it holds* $\|S' - f'\|_\infty = \mathcal{O}(h^\alpha)$ *(respectively,* $\|S' - f'\|_\infty = \mathcal{O}(h^{1+\alpha})$*).*

*Proof.* Let us recall from [1] the representation of histopolating spline

$$\begin{aligned} S(x) = z_i + h_i &\frac{m_{i-1}}{\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)^2} \log\left(\frac{m_{i-1}}{m_i}\right)^{1/2} \\ - h_i &\frac{m_{i-1}}{\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\left(1 + \frac{x - x_{i-1}}{h_i}\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\right)} \end{aligned}$$

for $x \in [x_{i-1}, x_i]$. This gives

$$S'(x) = \frac{m_{i-1}}{\left(1 + \frac{x - x_{i-1}}{h_i}\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\right)^2}.$$

First, let $f$ satisfy the assumptions of Lemma 1. We have found in its proof that

$$A = \left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1 = \frac{1}{2}\left(\frac{m_{i-1}}{m_i} - 1\right) + \mathcal{O}(h^{1+\alpha})$$

and we know that $A = \mathcal{O}(h^\alpha)$. Thus, we have

$$\begin{aligned} S'(x) &= \frac{m_{i-1}}{1 + 2\frac{x - x_{i-1}}{h_i}A + \mathcal{O}(h^{2\alpha})} = m_{i-1}\left(1 - 2\frac{x - x_{i-1}}{h_i}A + \mathcal{O}\left(h^{2\alpha}\right)\right) \\ &= m_{i-1} - \frac{x - x_{i-1}}{h_i}\frac{m_{i-1}}{m_i}\left(m_{i-1} - m_i\right) + \mathcal{O}\left(h^{2\alpha}\right). \end{aligned} \tag{4.1}$$

Using here the replacements $m_{i-1} = f'(x_{i-1}) + \mathcal{O}(h^\alpha)$ and

$$m_{i-1} - m_i = f'(x_{i-1}) + \mathcal{O}(h^\alpha) - (f'(x_i) + \mathcal{O}(h^\alpha)) = \mathcal{O}(h^\alpha)$$

together with $m_{i-1}, m_i \in [c_1, c_2]$ for some $c_1, c_2 > 0$, we obtain

$$S'(x) = f'(x_{i-1}) + \mathcal{O}(h^\alpha), \quad x \in [x_{i-1}, x_i]. \tag{4.2}$$

Obviously, $f'(x) = f'(x_{i-1}) + \mathcal{O}(h^\alpha)$, $x \in [x_{i-1}, x_i]$, and this with (4.2) gives one of the assertions of Lemma 3. Secondly, consider the case of $f$ satisfying the assumptions of Lemma 2. Now use in (4.1) the replacements

$$\begin{aligned} &m_{i-1} = f'(x_{i-1}) + \mathcal{O}(h^{1+\alpha}), \\ &\frac{m_{i-1}}{m_i} = \frac{f'(x_{i-1}) + \mathcal{O}(h^{1+\alpha})}{f'(x_i) + \mathcal{O}(h^{1+\alpha})} = \frac{f'(x_i) + \mathcal{O}(h)}{f'(x_i) + \mathcal{O}(h^{1+\alpha})} = 1 + \mathcal{O}(h), \\ &m_{i-1} - m_i = f'(x_{i-1}) + \mathcal{O}(h^{1+\alpha}) - \left(f'(x_i) + \mathcal{O}(h^{1+\alpha})\right) \\ &\qquad\qquad\qquad\qquad = -h_i f''(x_{i-1}) + \mathcal{O}\left(h^{1+\alpha}\right). \end{aligned}$$

Observe also that, at this time, $A = \mathcal{O}(h)$ and the remaining term in (4.1) is $\mathcal{O}(h^2)$. Then we have

$$S'(x) = f'(x_{i-1}) + (x - x_{i-1})f''(x_{i-1}) + \mathcal{O}(h^{1+\alpha}), \quad x \in [x_{i-1}, x_i].$$

This with the Taylor expansion

$$f'(x) = f'(x_{i-1}) + (x - x_{i-1})f''(x_{i-1}) + \mathcal{O}(h^{1+\alpha}), \quad x \in [x_{i-1}, x_i],$$

implies the other assertion of Lemma 3. The proof is complete. ■

We summarize the estimates of lemmas in the following theorem.

**Theorem 1.** *Suppose $f'(x) > 0$ for all $x \in [a, b]$ and $f' \in \operatorname{Lip}\alpha$ for some $\alpha \in (0, 1]$. Then the histopolating spline $S$ satisfies $\|S - f\|_\infty = \mathcal{O}(h^{1+\alpha})$. If, in addition, $f'' \in \operatorname{Lip}\alpha$, $\alpha \in (0, 1]$, then $\|S - f\|_\infty = \mathcal{O}(h^{2+\alpha})$.*

*Proof.* The histopolation condition

$$\frac{1}{h_i} \int_{x_{i-1}}^{x_i} S(x)\,dx = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

is equivalent to $\int_{x_{i-1}}^{x_i} (S(x) - f(x))\,dx = 0$, which implies the existence of $\xi_i \in (x_{i-1}, x_i)$ such that $S(\xi_i) = f(\xi_i)$. Therefore, it holds

$$S(x) - f(x) = \int_{\xi_i}^{x} (S'(s) - f'(s))\,ds.$$

Assuming $\|S' - f'\|_\infty \le Mh^\beta$ for some $M > 0$, we have for $x \in [x_{i-1}, x_i]$

$$\Big|S(x) - f(x)\Big| \le \Big| \int_{\xi_i}^{x} \Big|S'(s) - f'(s)\Big|\,ds\Big| \le Mh^\beta |x - \xi_i| \le Mh^{\beta+1}.$$

Basing now on Lemma 3 we get the assertion of Theorem 1. ■

## 5. Numerical Tests

We histopolated on the interval $[0, 1]$ the function $f(x) = \sin x$ to confirm the highest theoretical rate $\mathcal{O}(h^3)$ and also the piecewise quadratic function

$$f(x) = \begin{cases} -0.5x^2 + x & \text{for} \quad 0 \le x \le 0.5, \\ \phantom{-}0.5x^2 + 0.25 & \text{for} \quad 0.5 \le x \le 1, \end{cases}$$

having $f' \in \operatorname{Lip} 1$. However, the last function is such that $f'' \in \operatorname{Lip}\alpha$ does not hold for no one $\alpha \in (0, 1]$. Thus, here the rate $\mathcal{O}(h^2)$ coincides with those predicted by Theorem 1.

The mesh was nonuniform of the following form. Taking $h = 1/n$, central knots were calculated as

$$x_{\frac{n}{2}} = \frac{1+h}{2}, \ \ x_{\frac{n}{2}-1} = x_{\frac{n}{2}} - h, \ \ x_{\frac{n}{2}-2} = x_{\frac{n}{2}-1} - h, \ \ x_{\frac{n}{2}+1} = x_{\frac{n}{2}} + \frac{h}{10}.$$

Another ones were spaced uniformly on rest parts of the interval, i.e.

$$x_i = i x_{\frac{n}{2}-2}/(n/2-2), \qquad i = 1, \ldots, n/2-3,$$
$$x_{\frac{n}{2}+1+i} = x_{\frac{n}{2}+1} + i(1 - x_{\frac{n}{2}+1})/(n/2-1), \qquad i = 1, \ldots, n/2-2.$$

We used the boundary conditions (2.2) with $\alpha = f'(x_0)$ and $\beta = f'(x_n)$. The "tridiagonal" nonlinear system to determine the values of $m_i$ consisting of equations (2.5) was solved by Newton's method. The errors $\|S - f\|_\infty$ were calculated approximately on tenfold refined grid as

$$\varepsilon_n = \max_{1\le i\le n} \max_{0\le k\le 10} |(S-f)(x_{i-1} + kh_i/10)|.$$

Results of numerical tests are presented in Tables 1 and 2.

**Table 1.** Numerical results for $f(x) = \sin x$.

| $n$ | 8 | 16 | 32 | 64 | 128 |
|---|---|---|---|---|---|
| $\varepsilon_n$ | $1.15 \cdot 10^{-4}$ | $1.46 \cdot 10^{-5}$ | $1.84 \cdot 10^{-6}$ | $2.30 \cdot 10^{-7}$ | $2.87 \cdot 10^{-8}$ |
| $\varepsilon_{2n}/\varepsilon_n$ | | 7.874 | 7.961 | 7.988 | 7.996 |

**Table 2.** Numerical results for piecewise quadratic function.

| $n$ | 8 | 16 | 32 | 64 | 128 |
|---|---|---|---|---|---|
| $\varepsilon_n$ | $7.39 \cdot 10^{-4}$ | $1.79 \cdot 10^{-4}$ | $4.41 \cdot 10^{-5}$ | $1.09 \cdot 10^{-5}$ | $2.71 \cdot 10^{-6}$ |
| $\varepsilon_{2n}/\varepsilon_n$ | | 4.117 | 4.071 | 4.040 | 4.021 |

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