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DYNAMICS OF SMALL BUBBLE INTERFACE PERTURBATIONS IN VERTICAL HELE-SHAW CELL WITH MAGNETIC LIQUID UNDER THE ACTION OF NORMAL MAGNETIC FIELD

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Abstract. A linearized problem of dynamics for small perturbations of the gas bubble rising in the Hele-Shaw cell filled with magnetic liquid is considered. It is reduced to searching of eigenvalues and eigenfunctions for a linear operator with periodic boundary conditions. The obtained operator is presented as a sum of two linear operators: the second order differential operator with varying coefficients and the integro – differential operator with the singularity of the Cauchy type. The spectral problem is solved by the Degenerate Matrices (DM) method using Chebyshev polynomials of the first and second kind.

Key words: Hele-Shaw cell, gas bubble, magnetic liquid, eigenvalues and eigenfunctions problem, the Degenerate Matrices method, Chebyshev polynomials

1. Introduction

The free surface flows in Hele-Shaw cells have caused great interest recently [2, 18]. This is due to extremely rich pattern formation phenomena existing in these systems. Such phenomena obtain new features if one of the liquids in the Hele-Shaw cell is magnetic. In early eighties it was found that self-magnetic field forces cause the formation of the intricate labyrinthine patterns in those systems [6, 7]. Extension of those phenomena for the case of continuous energy supply was carried out in [5, 11, 12, 15], where the influence of the self-magnetic forces on the development of the Saffman-Taylor instability was considered both theoretically [5, 11] and experimentally [12, 15] for plane and circular geometries [5, 8]. Essential feature of these phenomena consists in the formation of the magnetic liquid finger, which dynamics and properties except some qualitative numerical experiments [7] remains poorly understood. This is not the case of the non-magnetic liquid fingers, which starting from pioneering work by Saffman and Taylor were extensively investigated [1, 10, 13, 14, 16, 17].

A related problem of the free interface dynamics describes the dynamics of the bubble rising in the vertical Hele-Shaw cell under action of the gravity [19]. Saffman and Taylor have found that for this case the family of the steady bubble shapes exists in the absence of the surface tension [19]. It is natural to assume that in the presence of surface tension the circular shape of the rising bubble is selected though its stability remain poorly understood. The preliminary numerical experiments [4] have shown that in the case of the bubble rising in the Hele-Shaw cell filled with magnetic liquid, the different families of rising bubble – pearlike or bent dumb-bell may arise. The stability of the rising bubble in the presence of magnetic forces is not investigated yet.

In the present paper we find numerically the eigenvalues and eigenvectors of the linear operator, which determines the evolution of small perturbations. By using the Degenerate Matrices (DM) method we have constructed two algorithms, that are based on the unsaturated approximation technique by Lagrange interpolations [8, 9]. The problem formulation and derivation of the operators in an explicit form are given in Section 2. Numerical approximation technique is described in Section 3. Numerical results and their discussion are given in Section 4.

2. Mathematical Formulation of the Problem

We consider a bubble in an infinite layer of the magnetic liquid. Let

$$\bar{r} = R\big(1 + \zeta(\alpha, t)\big)$$

be the equation of the free interface for a bubble in polar coordinates $(\bar{r}, \alpha)$ connected with its moving center, where $\zeta(\alpha, t)$ is the dimensionless interface perturbation of the circular bubble $\bar{r} = R$ in the Hele-Shaw cell. Taking the kinematic boundary condition from the Darcy equation accounting for the magnetic forces [3] and linearizing it with respect to ζ we obtain the following problem in the dimensionless form:

$$\frac{\partial \zeta}{\partial t} = -Bg \sin\alpha \frac{\partial \zeta}{\partial \alpha} - 2Bg\cos\alpha\,\zeta + \frac{\partial p}{\partial r}\Big|_{r=1}, \tag{2.1}$$

$$p|_{r=1} = -\frac{\partial^2 \zeta}{\partial \alpha^2} - (2Bg\cos\alpha + a)\zeta + \frac{Bm}{2h^2}\int_{-\pi}^{\pi}\Big[\frac{\zeta(\tau,t) - \zeta(\alpha,t)}{|\sin(\alpha-\tau)/2|} - \frac{\zeta(\tau,t)}{\sqrt{\sin^2(\alpha-\tau)/2 + h^2/4}}\Big]\,d\tau, \tag{2.2}$$

$$\Delta p = 0 \ \text{ if } r > 1, \quad p = c \ln r + \mathcal{O}(1) \ \text{ if } r \to +\infty\,. \tag{2.3}$$

Here $c = c(t)$ generally depends on t and it is determined by the equality

$$\int_{-\pi}^{\pi} \zeta(\tau, t)\, d\tau = 0 \tag{2.4}$$

and equations (2.1) and (2.3). After simple computations it can be shown that

$$c = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\tau\ \zeta(\tau, t)\, d\tau.$$

Initial value should be specified for ζ function at $t = 0$:

$$\zeta(\alpha, 0) = \zeta_0(\alpha)\,. \tag{2.5}$$

Here the following dimensionless parameters were introduced: $r = \frac{\bar{r}}{R}$ is the radius, $h = \frac{\bar{h}}{R}$ the thickness of the Hele–Shaw cell, Bm the magnetic Bond number [2, 3]; Bg the gravitational Bond number; $p = p(re^{i\alpha}, t)$ the effective pressure in the magnetic liquid outside the bubble. The constant a is determined as

$$a = 1 - \frac{2Bm}{h^2}\left(2 + \int_0^{\pi/2} \frac{\cos 2\tau\, d\tau}{\sqrt{\sin^2\tau + h^2/4}}\right), \tag{2.6}$$

or by elliptic functions

$$a = 1 - \frac{2Bm}{h^2}\Big(2 - \frac{2E}{k} + \Big(\frac{2}{k} - k\Big)K\Big)\,,$$

where

$$K = \int_0^{\pi/2} \frac{d\tau}{\sqrt{1 - k^2 \sin^2\tau}}, \qquad E = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\tau}\, d\tau$$

are full elliptic integrals of the first and second kind respectively, $k = \dfrac{1}{\sqrt{1 + h^2/4}}$.

Since coefficients of problem (2.1) – (2.5) do not depend explicitly on time, we represent the solution in the form

$$\zeta = e^{\lambda t} u(\alpha), \quad p|_{r=1} = e^{\lambda t}(\mathbf{P}u)(\alpha), \tag{2.7}$$

where $\mathbf{P}$ is a linear operator. This leads us to the following spectral problem in the set Ω of twice continuously differentiable 2π periodic functions:

$$-\lambda u = Bg \sin\alpha \frac{du}{d\alpha} + 2Bg\cos\alpha\, u - \frac{1}{4\pi}(\text{v.p.}) \int_{-\pi}^{\pi} \frac{(\mathbf{P}u)(\tau) - (\mathbf{P}u)(\alpha)}{\sin^2(\tau - \alpha)/2}\, d\tau$$
$$- \frac{Bm}{2\pi} \int_{-\pi}^{\pi} \cos\tau\ u(\tau)\, d\tau\,, \tag{2.8}$$

$$(\mathbf{P}u)(\alpha) = -\frac{d^2u}{d\alpha^2} - (a + 2Bg\cos\alpha)\,u - \frac{Bm}{2h^2}\int\limits_{-\pi}^{\pi}\frac{u(\tau)\,d\tau}{\sqrt{\sin^2(\tau-\alpha)/2 + h^2/4}} + \frac{Bm}{2h^2}\int_{-\pi}^{\pi}\frac{u(\tau)-u(\alpha)}{|\sin(\tau-\alpha)/2|}\,d\tau, \quad (2.9)$$

with the complementary condition

$$\int_{-\pi}^{\pi} u(\alpha)d\alpha = 0. \quad (2.10)$$

Proof. Equation (2.9) follows from (2.2) and the representations (2.7). A bounded harmonic function $p(re^{i\alpha})$ for $r > 1$ can be represented by the Schwarz integral. Differentiating p we get

$$\frac{\partial}{\partial r}p(re^{i\alpha}) = -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{(1+r^2)\cos(\alpha-\tau) - 2r}{\left(1 - 2r\cos(\alpha-\tau) + r^2\right)^2}\,p(e^{i\tau})\,d\tau,$$

which can be transformed for $r > 1$ as

$$\frac{\partial}{\partial r}p(re^{i\alpha}) = -\frac{1}{\pi}(\text{v.p.})\int_{-\pi}^{\pi}\frac{(1+r^2)\cos(\alpha-\tau) - 2r}{\left(1 - 2r\cos(\alpha-\tau) + r^2\right)^2}[p(e^{i\tau}) - p(e^{i\alpha})]\,d\tau, \quad (2.11)$$

since

$$\int_{-\pi}^{\pi}\frac{(1+r^2)\cos(\tau-\alpha) - 2r}{(1 - 2r\cos(\tau-\alpha) + r^2)^2}\,d\tau = 0\,.$$

The last equality can be obtained using the substitution $\xi = \exp\big(i(\alpha-\tau)\big)$ and residue's theory for complex integrals. Taking the limit $r \to 1{+}0$ in (2.11) gives

$$\frac{\partial p}{\partial r}\Big|_{r=1} = \frac{1}{4\pi}(\text{v.p.})\int_{-\pi}^{\pi}\frac{(\mathbf{P}u)(\tau) - (\mathbf{P}u)(\alpha)}{\sin^2(\tau-\alpha)/2}\,d\tau.$$

For unbounded harmonic functions satisfying (2.3) we must add a constant c to $\frac{\partial p}{\partial r}\big|_{r=1}$ in (2.1). This constant can be determined by integrating (2.1) and using (2.4). This gives us (2.8). ■

3. Numerical Solution of Problem (2.8) – (2.10)

In this section we present an efficient method for numerical solution of the spectral problem (2.8) – (2.10). It is based on nonsaturated approximations of eigenfunctions with the Chebyshev polynomials – analogous to how it was done in [8] for the Mathieu functions. The method is different for even and odd eigenfunctions $u^+(\alpha)$, $u^-(\alpha)$, where

$$u^+(\alpha) = \frac{u(\alpha) + u(-\alpha)}{2}, \quad u^-(\alpha) = \frac{u(\alpha) - u(-\alpha)}{2}\,.$$

Separation of even and odd parts of (2.8) – (2.10) gives two spectral problems:

$$-\lambda^{\pm} u^{\pm} = Bg \sin\alpha \, \frac{du^{\pm}}{d\alpha} + 2Bg\cos\alpha \, u^{\pm} - (\text{v.p.}) \int_{-\pi}^{\pi} \frac{(\mathbf{P}u^{\pm})(\tau) - (\mathbf{P}u^{\pm})(\alpha)}{4\pi\big(\sin^2(\tau-\alpha)/2\big)} \, d\tau - \frac{Bm}{2\pi} \int_{-\pi}^{\pi} \cos\tau \, u^{\pm}(\tau) \, d\tau, \tag{3.1}$$

$$(\mathbf{P}u^{\pm})(\alpha) = -\frac{d^2 u^{\pm}}{d\alpha^2} - (a + 2Bg\cos\alpha) u^{\pm} - \frac{Bm}{2h^2} \int_{-\pi}^{\pi} \frac{u^{\pm}(\tau)\, d\tau}{\sqrt{\sin^2(\tau-\alpha)/2 + h^2/4}} + \frac{Bm}{2h^2} \int_{-\pi}^{\pi} \frac{u^{\pm}(\tau) - u^{\pm}(\alpha)}{|\sin(\tau-\alpha)/2|} \, d\tau, \tag{3.2}$$

with the complementary condition

$$\int_{-\pi}^{\pi} u^{\pm}(\alpha)\, d\alpha = 0. \tag{3.3}$$

In the case of even eigenfunctions we use Chebyshev polynomials of the first kind, but in the case of odd eigenfunctions, Chebyshev polynomials of the second kind are used. As a result we obtain two eigenvalue and eigenvector problems for $N \times N$ matrices $\mathbf{A}_N^{\pm}$. The approximate solutions of (2.8) – (2.10) can be represented by means of the solutions of spectral problems for these matrices. In general, the accuracy of approximations increases when N is increased.

Proposition 1. *A computation of even eigenfunctions $u^{+}(\alpha)$ leads to the following spectral problem*

$$(\mathbf{A}_N^{+} + \lambda^{+}\mathbf{E}_N)\vec{w} = 0, \quad \vec{w} = (w_1, w_2, \ldots, w_N)^T, \tag{3.4}$$

$$\mathbf{A}_N^{+} = Bg\big[\{\text{diag}(x_k^2 - 1)\}\mathbf{\Delta}_N^{(1)} + 2\{\text{diag}\, x_k\}\big] - \mathbf{G}_N^{+}\mathbf{P}_N,$$

$$\mathbf{G}_N^{+} = \frac{1}{2N}\big[2\mathbf{H}_N^{+} - \{\text{diag}(x_k^2-1)\}(\mathbf{\Delta}_N^{(1)})^2 - \{\text{diag}\, x_k\}\mathbf{\Delta}_N^{(1)}\big],$$

$$\mathbf{P}_N = \{\text{diag}(x_k^2-1)\}(\mathbf{\Delta}_N^{(1)})^2 + \{\text{diag}\, x_k\}\mathbf{\Delta}_N^{(1)} - \{\text{diag}(a + 2Bg\, x_k)\} + \frac{Bm}{h^2}\Big\{-4\{\text{diag}\, x_k\}\mathbf{\Delta}_N^{(1)} + \frac{\pi}{N}[\mathbf{M}_N^{+} - \{\text{diag}\, s_k^{+}\}\mathbf{\Delta}_N^{(1)}]\Big\}.$$

Here $\{\text{diag}\, \mu_k\}$ is a diagonal matrix with elements μ_k, $k = 1, 2, \ldots, N$,

$$x_k = -\cos\frac{(2k-1)\pi}{2N}, \quad k = 1, 2, \ldots, N,$$

$$s_k^{+} = \sum_{j=1, j\neq k}^{N} (x_j - x_k)\Phi(x_k, x_j, 0),$$

$$\Phi(x,\xi,h) = \left[\frac{1-x\xi+h^2/2+\sqrt{(\xi-x)^2+(1-x\xi)h^2+h^4/4}}{(\xi-x)^2+(1-x\xi)h^2+h^4/4}\right]^{1/2}, \tag{3.5}$$

$\mathbf{\Delta}_N^{(1)}$ is the Chebyshev differentiation matrix with elements

$$\delta_{kj}^{(1)} = \frac{T_N'(x_k)}{(x_k-x_j)T_N'(x_j)}, \text{ if } j \neq k, \quad \delta_{kk}^{(1)} = \frac{T_N''(x_k)}{2T_N'(x_k)},$$

$\mathbf{H}_N^+$ and $\mathbf{M}_N^+$ are matrices with elements

$$h_{kj}^+ = \frac{1-x_jx_k}{(x_j-x_k)^2}, \text{ if } j \neq k, \quad h_{kk}^+ = \frac{x_k^2}{4(1-x_k^2)} - \frac{N^2-1}{3},$$

$$m_{kj}^+ = \Phi(x_k,x_j,0) - \Phi(x_k,x_j,h), \text{ if } j \neq k,$$

$$m_{kk}^+ = -\Phi(x_k,x_k,h) - \sum_{j=1,j\neq k}^{N} \Phi(x_k,x_j,0).$$

Let λ_m^+ and $\vec{w}(m) = (w_1(m),\dots,w_N(m))^T, m = 1,2,\dots,N$, be the eigenvalues and corresponding eigenvectors of (3.4). Then the approximate eigenfunction $u_m^+(\alpha)$ corresponding to λ_m^+ can be represented as follows

$$u_m^+(\arccos x) \approx T_N(x)\sum_{k=1}^{N}\frac{w_k(m)}{(x-x_k)T_N'(x_k)}, \ x = \cos\alpha, \tag{3.6}$$

or

$$u_m^+(\alpha) \approx \cos N\alpha \sum_{k=1}^{N}\frac{(-1)^{N+k}w_k(m)\sin[(2k-1)\pi/(2N)]}{\cos\alpha+\cos[(2k-1)\pi/(2N)]}. \tag{3.7}$$

Proof. We introduce new variables $x = \cos\alpha$, $\xi = \cos\tau$ in (3.1) – (3.3) for λ^+; $u^+(\alpha)$, and observe that rapidly convergent series for the eigenfunction $u^+(\alpha)$ according to $\{\cos k\alpha\}, k = 0,1,\dots$ transforms into series with respect to $\{T_k(x)\}$. Denoting $u^+(\arccos x) = u(x)$ and after some elementary manipulations, we obtain

$$-\lambda^+u(x) = Bg[(x^2-1)u'(x)+2xu(x)] - (v.p.)\frac{1}{\pi}\int_{-1}^{1}\frac{[(\tilde{\mathbf{P}}u)(\xi)-(\tilde{\mathbf{P}}u)(x)](1-x\xi)d\xi}{\sqrt{1-\xi^2}\,(\xi-x)^2}. \tag{3.8}$$

The linear operator $\tilde{\mathbf{P}}$ is defined by the equality $(\mathbf{P}u)(\arccos x) = (\tilde{\mathbf{P}}u)(x)$. Therefore,

$$(\tilde{\mathbf{P}}u)(x) = (x^2-1)u''(x) + xu'(x) - (a+2Bg\,x)u(x) + \frac{Bm}{h^2}\int_{-1}^{1}\frac{[u(\xi)-u(x)]\Phi(x,\xi,0)-u(\xi)\Phi(x,\xi,h)}{\sqrt{1-\xi^2}}\,d\xi, \tag{3.9}$$

where $\Phi(x,\xi,h)$ is defined by (3.5). Since

$$(v.p.)\int_{-1}^{1}\frac{d\xi}{(\xi-x)\sqrt{1-\xi^2}}=0\,,$$

we have

$$(v.p.)\int\limits_{-1}^{1}\frac{[(\tilde{\mathbf{P}}u)(\xi)-(\tilde{\mathbf{P}}u)(x)(1-x\xi)}{\sqrt{1-\xi^2}\,(\xi-x)^2}\,d\xi=-x\int\limits_{-1}^{1}\frac{(\tilde{\mathbf{P}}u)(\xi)-(\tilde{\mathbf{P}}u)(x)}{\sqrt{1-\xi^2}\,(\xi-x)^2}\,d\xi$$
$$+(1-x^2)\int_{-1}^{1}\frac{(\tilde{\mathbf{P}}u)(\xi)-(\tilde{\mathbf{P}}u)(x)-(\xi-x)(\tilde{\mathbf{P}}u)'(x)}{\sqrt{1-\xi^2}\,(\xi-x)^2}\,d\xi.$$

On the other hand

$$\int_{1}^{1}\frac{u(\xi)-u(x)}{\sqrt{1-\xi^2}}\,\Phi(x,\xi,0)\,d\xi=-4xu'(x)$$
$$+\int_{-1}^{1}\frac{u(\xi)-u(x)-(\xi-x)u'(x)}{\sqrt{1-\xi^2}}\,\Phi(x,\xi,0)\,d\xi\,.$$

Evaluating (3.8) – (3.9) at x_k, which are zeroes of the Chebyshev polynomial $T_N(x)$, evaluating all derivatives according to the differentiation matrix $\mathbf{\Delta}_N^{(1)}$, and approximating integrals by the following quadrature formula

$$\int_{-1}^{1}\frac{f(\xi)}{\sqrt{1-\xi^2}}\,d\xi\approx\frac{\pi}{N}\sum_{j=1}^{N}f(-\cos[(2j-1)\pi/(2N)]),\qquad(3.10)$$

we obtain the eigenvalue and eigenvector problem for matrix $\mathbf{A}_N^+$. Each eigenvalue λ_m^+ corresponds to the eigenvector $\vec{w}(m)$, which have the components

$$w_k(m)\approx u_m^+(\arccos x_k),\quad k=1,\ldots,N\,.$$

Therefore, formulas (3.6) and (3.7) are obtained as the Lagrange interpolations. ■

Proposition 2. *A computation of odd eigenfunctions $u^-(\alpha)$ leads to the following spectral problem for matrices*

$$(\mathbf{A}_N^-+\lambda^-\mathbf{E}_N)\vec{v}=0,\quad \vec{v}=(v_1,v_2,\ldots,v_N)^T,\qquad(3.11)$$

$$\mathbf{A}_N^-=Bg\left(\{\mathrm{diag}(z_k^2-1)\}\mathbf{\Delta}_N^{(2)}+3\{\mathrm{diag}\ z_k\}\right)-\mathbf{G}_N^-\mathbf{Q}_N,$$

$$\mathbf{G}_N^-=\frac{1}{N+1}\left[\mathbf{H}_N^--\frac{1}{2}\{\mathrm{diag}(z_k^2-1)\}(\mathbf{\Delta}_N^{(2)})^2-(N-\frac{7}{2})\{\mathrm{diag}\ z_k\}\mathbf{\Delta}_N^{(2)}\right],$$

$$\mathbf{Q}_N=\{\mathrm{diag}(z_k^2-1)\}(\mathbf{\Delta}_N^{(2)})^2+[3-4Bm/(3h^2)]\{\mathrm{diag}\ z_k\}\mathbf{\Delta}_N^{(2)}$$
$$-\{\mathrm{diag}\Big(2Bg\,z_k+a-1-\frac{4Bm}{h^2}\Big)\}+\frac{\pi}{N+1}\Big[-\{\mathrm{diag}\ s_k^-\}\mathbf{\Delta}_N^{(2)}+\mathbf{M}_N^-\Big].$$

Here a is calculated from (2.6),

$$z_k = -\cos\Big(\frac{k\pi}{N+1}\Big), \quad k = 1, 2, \ldots, N$$

are zeroes of the Chebyshev polynomial of the second kind $U_N(x)$, $\mathbf{\Delta}_N^{(2)}$ is Chebyshev differentiation matrix with elements

$$\delta_{kj}^{(2)} = \frac{U_N'(z_k)}{(z_k - z_j)U_N'(z_j)}, \text{ if } \ j \neq k, \quad \delta_{kk}^{(2)} = \frac{U_N''(z_k)}{2U_N'(z_k)},$$

$\mathbf{H}_N^-$ and $\mathbf{M}_N^-$ are matrices with elements

$$h_{kj}^- = \frac{1 - z_j^2}{(z_j - z_k)^2}, \text{ if } j \neq k, \quad h_{kk}^- = -\frac{z_k}{4(1 - z_k^2)} - \frac{N^2 + 2N + 3}{3},$$

$$m_{kj}^- = (1 - z_j^2)[\Psi(z_k, z_j, 0) - \Psi(z_k, z_j, h)], \text{ if } j \neq k;$$

$$m_{kk}^- = -(1 - z_k^2)\Psi(z_k, z_k, h) - \sum_{j=1, j\neq k}^{N} (1 - z_j^2)\Psi(z_k, z_j, h).$$

Function Ψ is defined as

$$\Psi(x, \xi, h) = g(x, \xi, h)\Big[1 - x\xi + \frac{h^2}{2} + g(x, \xi, h)\Big]^{1/2},$$

where $g(x, \xi, h) = [(\xi - x)^2 + (1 - x\xi)h^2 + h^4/4]^{1/2}$,

$$s_k^- = \sum_{j=1, j\neq k}^{N} (1 - z_j^2)(z_j - z_k)\Psi(z_k, z_j, 0).$$

Here λ_m^- and $\vec{v}(m) = (v_1(m), \ldots, v_N(m))^T, m = 1, 2, \ldots, N$, are the eigenvalues and corresponding eigenvectors of problem (3.11). The approximation of eigenfunction $u_m^-(\alpha)$ can be represented as:

$$u_m^-(\arccos x) \approx \sqrt{1 - x^2}\, U_N(x) \sum_{k=1}^{N} \frac{v_k(m)}{(x - z_k)U_N'(z_k)}, \quad x = \cos\alpha, \qquad (3.12)$$

where $z_k = -\cos[k\pi/(N+1)]$, or

$$u_m^-(\alpha) \approx \frac{\sin{(N+1)\alpha}}{N+1} \sum_{k=1}^{N} \frac{(-1)^{N+k+1} v_k(m) \sin[k\pi/(N+1)]}{\cos\alpha + \cos[k\pi/(N+1)]}. \qquad (3.13)$$

Proof. A proof of this case is analogous with the case of even eigenfunctions. Only now for λ^- and $u^-(\alpha)$ in (3.1), (3.2) we must use the substitutions of variables

$$x = \cos\alpha, \ \xi = \cos\tau, \ u^-(\alpha)/\sin\alpha = v(\cos\alpha). \qquad (3.14)$$

Instead of the linear operator **P** we obtain the linear operator **Q**, which is defined by equality $(\mathbf{P}u^{-})(\alpha) = \sqrt{1-x^2}(\mathbf{Q}v)(x)$ and substitutions (3.14). Odd eigenfunctions can be represented by rapidly convergent series with respect to $\{\sin k\alpha\},\ k = 1, 2, \ldots$. Since

$$\frac{\sin[(n+1)x]}{\sin x} = U_n(x), \quad n = 0, 1, \ldots,$$

we obtain (3.12) – (3.13) by using substitutions in (3.1), (3.2) and evaluating it for zeroes of the Chebyshev polynomial of the second kind $U_N(x)$. ■

It is difficult to estimate theoretically the error of the presented numerical methods. Therefore, we judge about the errors by means of the following two procedures:

1. The problem is solved with different N.

2. In the case $Bg = 0$, numerical results are compared with the exact solution:

$$\lambda_m^+ = \lambda_m^- = -m\,\varphi(m), \quad u_m^+(\alpha) = C_1 \cos m\alpha, \quad u_m^-(\alpha) = C_2 \sin m\alpha\,, \quad (3.15)$$

where

$$\varphi(m) = m^2 - 1 + \frac{4Bm}{h^2}\Big[1 + \int_0^{\pi/2} \frac{\sin[(m+1)\tau]\sin[(m-1)\tau)]\;d\tau}{\sqrt{\sin^2\tau + h^2/4}} - \int_0^{\pi/2} \frac{\sin^2 m\tau}{\sin\tau}\,d\tau\Big], \quad (3.16)$$

C_1 and C_2 are arbitrary constants. A validity of (3.15) and (3.16) can be verified by substitution into (3.1) – (3.2).

4. Numerical Results

For the brevity, we present only numerical approximations of the eigenvalues of the spectral problem (2.8) – (2.10). Numerical methods given in the previous section allow us to compute the corresponding eigenfunctions also and then to calculate the solution of (2.1) – (2.5) by manipulations typically used for the Fourier method.

If $Bm = Bg = 0$ then it follows from (3.16) that

$$\lambda_m^+ = \lambda_m^- = -m(m^2-1)\,, \quad m \geq 1\,.$$

The presented numerical Degenerate Matrix (DM) method, which is based on nonsaturated approximations of eigenfunctions with Chebyshev polynomials, gives exact results in this case.

Positive eigenvalues are the most interesting for applications since the instability in the interface dynamics of the rising gas bubble can develop

Table 1. First eigenvalues of (2.8) – (2.10) for $Bg = 0,\ Bm = 5,\ h = 1$, which are obtained from (3.16) and by the DM method with $N = 20$ and $N = 80$.

		$N = 20$		$N = 80$	
m	(3.16)	λ_m^+	λ_m^-	λ_m^+	λ_m^-
1	0.00000	0.00000	0.00000	0.00000	0.00000
2	1.73746	1.73746	1.73746	1.73746	1.73746
3	-2.70893	-2.70893	-2.70893	-2.70893	-2.70893
4	-21.14669	-21.14690	-21.14720	-21.14669	-21.14669
5	-60.71987	-60.72118	-60.72271	-60.71988	-60.71988
6	-128.11744	-128.12218	-128.12677	-128.11746	-128.11748
7	-229.76718	-229.78019	-229.79090	-229.76723	-229.76727
8	-371.95106	-371.98103	-372.00245	-371.95117	-371.95125
9	-560.86741	-560.96723	-560.87106	-560.86764	-560.86778
10	-802.66397	-802.77857	-802.84284	-802.66439	-802.66464

on corresponding eigenforms. On the other hand, each eigenvalue becomes positive for sufficiently large value of Bm. This conclusion follows from (3.16).

In Table 1 we present the first ten eigenvalues of the problem (2.8) – (2.10), $Bg = 0,\ Bm = 5,\ h = 1$, which are calculated by (3.16) and DM methods with orders of the matrices $N = 20,\ N = 80$. As we see, the maximal relative error of the eigenvalues calculated by DM method is of the order 10^{-4} for $N = 20$ and 10^{-6} for $N = 80$.

Table 2. First eigenvalues of (2.8) – (2.10) for $Bg = Bm = 5,\ h = 1$, which are obtained by DM methods with $N = 20,\ N = 40$ and $N = 80$.

	$N = 20$		$N = 40$		$N = 80$	
m	λ_m^+	λ_m^-	λ_m^+	λ_m^-	λ_m^+	λ_m^-
1	1.02788	1.02783	1.02788	1.02787	1.02788	1.02788
2	10.70114	10.70112	10.70109	10.70109	10.70109	10.70109
3	-4.83183	-4.83191	-4.83179	-4.83180	-4.83179	-4.83179
4	-22.11347	-22.11388	-22.11321	-22.11323	-22.11319	-22.11319
5	-61.14637	-61.14801	-61.14504	-61.14514	-61.14496	-61.14496
6	-128.33369	-128.33839	-128.32910	-128.32939	-128.32882	-128.32884
7	-229.89494	-229.90576	-229.88254	-229.88321	-229.88179	-229.88183
8	-372.04665	-372.06819	-372.01824	-372.01959	-372.01655	-372.01663
9	-560.96721	-561.06589	-560.90932	-560.91173	-560.90589	-560.90603
10	-802.80129	-802.86569	-802.69314	-802.69715	-802.68676	-802.68701

In Table 2 we present the first ten eigenvalues of of the problem (2.8) – (2.10), $Bg = 5,\ Bm = 1,\ h = 1$, which are calculated by the DM method with orders of the matrices $N = 20,\ N = 40$ and $N = 80$. Comparing results obtained for λ^+ and λ^-, we see that the accuracy for $N = 20$ and $N = 80$ is the same as in the Table 1 for $Bg = 0$.

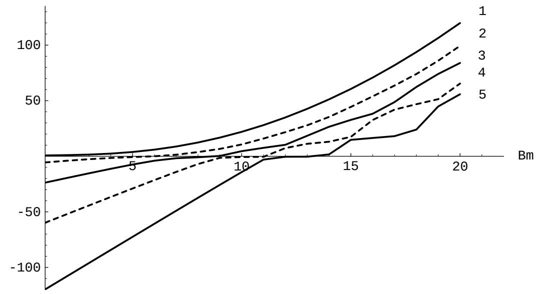

Figure 1. The curves of the first five eigenvalues: $\lambda = \lambda(Bm)$ for $Bg = 3$.

In Figure 1 the dependence of the first five eigenvalues on the magnetic Bond number Bm is shown (here $Bg = 3$).

5. Conclusions

The mathematical model for the dynamics of the gas bubble rising in the vertical Hele-Shaw cell filled with magnetic liquid is formulated as a spectral problem for the linear integro-differential operator with periodic boundary conditions. The obtained spectral problem is solved by the Degenerate Matrices methods, which are constructed by using differentiation matrices for the Lagrange projector with Chebyshev nodes. The accuracy of numerical calculations can be controlled easily by varying the order N of differentiation matrices. Results of numerical experiments are presented and they prove the efficiency of the developed method.

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Burbulo paviršiaus mažų žadinimų dinamika vertikalioje Hele-Shaw ląstelėje, užpildytoje magnetiniu skysčiu veikiamo normaliniu magnetiniu lauku

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Dujų burbulo, judančio vertikalią Hele-Shaw ląstelę užpildančiu magnetiniu skysčiu, paviršiaus dinamikos matematinis modelis yra suformuluotas kaip spektrinis uždavinys tam tikram tiesiniam operatoriui su periodinėmis kraštinėmis sąlygomis. Pastarasis uždavinys yra išspręstas skaitmeniškai išsigimstančių matricų metodu.