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SOME PROPERTIES OF FRACTIONAL BURGERS EQUATION

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ABSTRACT

The fractional generalization of a one-dimensional Burgers equation

$$\phi_t + \frac{1}{2} {}_aD_x^p \left({}_aD_x^{1-p} \phi \right)^2 - \alpha \phi_{xx} = 0 \,,$$

with initial conditions $\phi(x,0) = \phi_o(x)$; $\phi_t(x,0) = \psi_o(x)$, where $\phi = \phi(x,t) \in C^2(\Omega)$; $\phi_t \equiv \partial\phi/\partial t$; ${}_aD_x^p$ is the Riemann-Liouville fractional derivative of the order p; $\Omega = (x,t)$: $x \in E^1, t > 0$; and the explicit form of a particular analytical solution are suggested. Existing of traveling wave solution and conservation laws are considered. The relation with Burgers equation of integer order and properties of fractional generalization of the Hopf-Cole transformation are discussed.

1. INTRODUCTION

In the case when the properties of a system in a certain point of configuration or phase space depend not only on the properties of this system at this point, but also on the properties of at least one point of the environment, we deal with the nonlocal phenomena. As the examples, let us mention the well known prey–predator system of Volterra with delay in ecology [19], the ferromagnetic properties of matter in physics [3], viscoelastic phenomena in mechanics [12]. From the mathematical point of view, such phenomena usually are described by the integro-differential equations [2]. Over the last few years more attention has been given to a special part of the theory of integro-differential equations, the so-called fractional calculus [15; 16; 17]. This approach is applied not only in the theory of fractals and to the above-mentioned, already classical, nonlocal phenomena, but also for description of the electrical, biological and diffusion phenomena. The latter topic, as follows from the growing number of publications, receives the bulk of attention [16].

There are two fractional generalizations of the classical diffusion equation. One of them leads to replacing the second space derivative by the fractional one:

$$\phi_t - \alpha_a D_x^{2+p}\phi = 0\,, \tag{1.1}$$

where $\phi = \phi(x,t)$, ${}_aD_x^p$ is a fractional derivative in the sense of Riemann–Liouville [17]:

$${}_aD_x^p\phi(x,t) = \frac{1}{\Gamma(-p)}\int_a^x \frac{\phi(\xi,t)}{(x-\xi)^{p+1}}\,d\xi\,, \tag{1.2}$$

where $0 < p < 1$, and a is a parameter of nonlocality. From this approach, but for the wave equation, the fractional derivative was considered by A. N. Gerasimov in [6]. The other generalization, proposed by R. R. Nigmatullin in [14], is related to the fractional substitution of the time derivative:

$${}_aD_t^p\phi - \alpha\phi_{xx} = 0\,. \tag{1.3}$$

In this situation, due to the well known relation between the diffusion and the Burgers equations, we may expect two analogous nonlocal generalizations of Burgers equation:

$$\phi_t + \phi\phi_x - \alpha_a D_x^p\phi = 0\,, \tag{1.4}$$

and the other one

$${}_aD_t^p\phi + \phi\phi_x - \alpha\phi_{xx} = 0\,. \tag{1.5}$$

These generalizations (1.4) and (1.5) are just applications of the ideas of [6], [14]. However, most interesting and perhaps most productive is the third, fractional Burgers equation (FBE), i.e. nonlinear and nonlocal generalization of diffusion equation, based on a fractional generalization of the Hopf and Cole transformation and presented below:

$$\phi_t + \frac{1}{2}\,{}_aD_x^p\left({}_aD_x^{1-p}\phi\right)^2 - \alpha\phi_{xx} = 0\,, \tag{1.6}$$

in which $\phi(x,t)$, $\phi_0(x) \in \mathbb{R}, -\infty < x < +\infty\,;\, t \geq 0$ and the parameter $\alpha > 0$. A series of exact analytical solutions of this equation, the asymptotic form of the solutions and a fractional generalization of Reynolds number are presented. Concrete examples, which correspond to the simplest behavior of fractal solution, are analyzed.

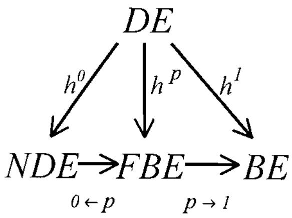

Figure 1. Interrelation among diffusion equation (DE), Burgers equation (BE) and fractional Burgers equation (FBE). The fractional parameter $0 \leq p \leq 1$ can take integer or fractional values.

2. THE REYNOLDS NUMBER AND NONLOCAL PERTURBATIONS

The acoustic Reynolds number is a very convenient dimensionless quantity which is used in the nonlinear Burgers equation (BE). This number is just a ratio of the nonlinear and the dissipative terms $\mathrm{R_a} \sim \frac{\phi\phi_x}{\alpha\phi_{xx}}$. In the case when $\mathrm{R_a} \ll 1$, the influence of the nonlinear term is negligible, but for $\mathrm{R_a} \gg 1$ this term plays a crucial role and leads to turbulence.

For the FBE (1.6) we may introduce a dimensionless generalization of the acoustic Reynolds number

$$\mathrm{R_a} \sim \frac{{}_aD_x^p \left({}_aD_x^{1-p}\phi\right)^2}{\alpha\phi_{xx}} \sim \frac{\phi x^p}{\alpha}\,. \tag{2.1}$$

Depending on the value of this number, we obtain two limit cases of FBE:

$$\phi_t - \alpha\phi_{xx} = 0\,, \quad \mathrm{R_a} \ll 1\,; \tag{2.2}$$

$$\phi_t + \frac{1}{2}\,{}_aD_x^p \left({}_aD_x^{1-p}\phi\right)^2 = 0\,, \quad \mathrm{R_a} \gg 1\,. \tag{2.3}$$

We shall see below that the Reynolds number (2.1) expresses a number of degrees of freedom of single perturbation and plays a no less important role in the nonlinear and nonlocal phenomena, similarly to the usual Reynolds number in hydrodynamics and acoustics.

3. SOLUTIONS OF THE FRACTIONAL BURGERS EQUATION

Note that in the case when $\mathrm{R_a} \ll 1$, the influence of the diffusion term is minimal and the FBE (1.6) turns into an ordinary diffusion equation. This implies a deep correlation between the DE and the FBE.

Suppose the solution of the diffusion equation $w(x,t)$ is known. Then, due to the useful relation between BE and DE at $p = 1$, we can express the solution of FBE (1.6) for any $p>0$:

$$\phi(x,t) = -2\alpha\, {}_aD_x^p \log w(x,t)\,. \tag{3.1}$$

This relation can be represented by a diagram (Fig. 1).

In the case of the solution of the diffusion equation

$$w(x,t) = \exp\left(-\frac{cx}{2\alpha} + \frac{c^2 t}{4\alpha} - b\right), \tag{3.2}$$

the solution of the FBE (1.6) is

$$\phi(x,t) = cx - \frac{c^2}{2}t + \frac{b}{2\alpha}\,, \qquad p = 0\,, \tag{3.3}$$

$$\phi(x,t) = c\,, \qquad p = 1\,, \tag{3.4}$$

$$\phi(x,t) = -\frac{1}{\Gamma(-p)}\frac{b(p-1) + \alpha c\,[2ap + (1-p)ct - 2x]}{2\alpha(p-1)p(x-a)^p}\,, \quad 0<p<1\,, \tag{3.5}$$

with asymptotic behavior the solution (3.5):

$$\phi(x,t) = \frac{(x-a)^{(1-p)}}{\Gamma(2-p)} \qquad \text{when} \quad x \to +\infty\,. \tag{3.6}$$

Note that solution (3.5) continuously transforms from solution (3.3) into ((3.4)) when the parameter p runs from $p=0$ to $p=1$. In general, despite the validity of the FBE (1.6) at the any $p>0$, the smooth dependence on p between integer and fractional values of p is only at $p<0$. When $p>0$ and $p \notin \mathbb{N}$ the $\Gamma(-p)$ exists and we can apply the analytical continuation. At the $p \in \mathbb{N}$ we have use the expression (3.1). Note also, that to the solution (3.2) of the usual diffusion equation we have not nontrivial solutions at the $p \geqslant 2$, when $p \in \mathbb{N}$.

Now consider a special type of the traveling wave solution when $\phi(x,t) = \phi(x-ut) \equiv \phi(\xi)$, where $\xi \equiv x - ut$. Then the FBE takes the form

$$\frac{1}{2}\int {}_aD_\xi^p \left[{}_aD_\xi^{(1-p)}\phi\right]^2 d\xi = \alpha\phi' + u\phi\,. \tag{3.7}$$

The traveling wave solution of (3.7) is

$$\phi = 2u\xi - 2\alpha\log\left[\exp\frac{u(c+\xi)}{\alpha} - 1\right], \quad p=0\,,$$

with the asymptotics $\phi = -2uc\,, \quad \xi \gg 1\,;$

$$\phi = -\frac{2}{\Gamma(2-p)}\frac{u\xi + \frac{p}{\alpha}u^2 - (1-p)C}{(\xi-a)^p}\,, \quad p<0\,,$$

with the integration constant C, and

$$\phi = \phi_1 + \frac{\phi_2 - \phi_1}{1 + \exp\left(\frac{\phi_2 - \phi_1}{2\alpha}\xi\right)}\,, \quad p = 1\,,$$

with the asymptotics $\phi(\xi \to +\infty) = \phi_1\,, \quad \phi(\xi \to -\infty) = \phi_2\,, \quad \phi_2 > \phi_1\,.$

The relation (3.1) between the solutions of FBE and DE allows to consider an interaction of nonlocal and nonlinear perturbations. Two or more perturbations moving with a different velocity can overtake each other or flow together into a new intensive perturbation. The FBE also describes the interaction process of two or more moving perturbations. The superposition principle is not valid for the nonlinear FBE, but it is valid for the linear diffusion equation. The fractional Hopf–Cole transformation (3.1) interrelates the solutions of nonlocal and nonlinear FBE and of linear diffusion equation. Thus, if $w_i(x,t)$ are the solutions of diffusion equation, then $\phi(x,t) = -2\alpha\,{}_aD_x^p(\log\sum w_i)$ are the solutions of FBE.

For two solutions of the diffusion equation

$$w_i(x,t) = a_i \exp\left(-\frac{c_i x}{2\alpha} + \frac{c_i^2 t}{4\alpha} - b_i\right), \quad i = 1,2$$

we obtain a nonlocal and nonlinear interaction of these perturbations

$$-\frac{\phi(x,t)}{2\alpha} = \begin{cases} \log(w_1 + w_2)\,, & p = 0\,, \\ \frac{1}{\Gamma(-p)}\int_a^x \frac{\log[w_1(\xi,t) + w_2(\xi,t)]}{(x-\xi)^{(p+1)}}\,d\xi\,, & 0 < p\,, \\ \frac{c_1 w_1 + c_2 w_2}{w_1 + w_2}\,, & p = 1\,. \end{cases}$$

4. THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS AND THE CONSERVATION LAWS

Despite the nonlocal character of FBE there are few conservation laws. Suppose that for some q

$${}_aD_x^{-q}\phi_{xx} \to 0\,, \quad {}_aD_x^{-p}\phi_x \to 0\,, \quad \text{when} \quad x \to a\,, \quad \text{and} \quad x \to +\infty\,.$$

Then, by acting on equation (1.6) with the nonlocal operator ${}_aD_x^{-q}$, we obtain

$$\frac{\partial}{\partial t}{}_aD_x^{-q}\phi = {}_aD_x^{-q}\,{}_aD_x^{p}\left[\alpha\,{}_aD_x^{2-p}\phi - \frac{1}{2}\left({}_aD_x^{1-p}\phi\right)^2\right].$$

Note here that for $p, q > 0$, and $k - 1 \le p < k$

$${}_aD_x^{-q}\,{}_aD_x^{p}f(x) = {}_aD_x^{p-q}f(x) - \sum_{j=1}^{k}\left[{}_aD_x^{p-j}f(x)\right]_{x=a}\frac{(x-a)^{q-j}}{\Gamma(1+q-j)}\,.$$

This means that

$$ {}_aD_x^{-q}\phi(x,t) = \frac{1}{\Gamma(p)}\int_a^x \frac{\phi(\xi,t)}{(x-a)^{1-q}}\,d\xi = inv = I\,. \qquad (4.1)$$

At the $q = p$, the conserved quantity I corresponds to the functional of length of the fractional curve $\phi(\xi)$.

Remind that the field function $\phi(x,t)$ has a fractional physics dimension $[\phi] = L^{2-p}T^{-1}$, $[\alpha] = L^2T^{-1}$. Therefore, at the $q = p$, we may consider the quantity I (4.1) as an expression of the impulse conservation law, which coincides with the well-known relation to the usual BE at $p = 1$.

Thus we arrive to an important conclusion: the dimensionless Reynolds number $\mathrm{R_a} \sim \frac{2I}{\alpha}$ has the same form for any fractional or integer meaning of the parameter p. The kinetic perturbation energy K in the fractional case can also be modified to $K = \frac{1}{2}\int_b^a \left({}_aD_x^{1-p}\phi\right)^2\,dx$.

It is easy to check that the kinetic energy K at the $b \to +\infty$ is a conserved quantity, if and only if the field function $\phi(x,t)$ obeys the equation (1.6):

$$\int_a^b \left[\frac{1}{2}\phi_{x\,\alpha}D_x^p({}_\alpha D_x^{1-p}\phi)^2 - \alpha\phi_x^2\right]dx > 0.$$

The conservation laws are obviously related to the group of automorphisms of FBE. In the case of the usual BE there is a large Lee symmetry group of point transformations, which contains the Galilei, dilation and projective transformations and is generated by operators

$$D = 2t\partial_t + x\partial_x - \phi\partial_\phi\,, \quad K = t^2\partial_t + tx\partial_x - \left(tu + \frac{1}{2}x\right)\partial_\phi\,,$$
$$P_1 = \partial_x\,, \quad P_2 = \partial_t\,, \quad B = 2t\partial_x - \partial_\phi\,.$$

Symmetries in the discrete BE have been studied in [8].

An important and in general case un-investigated problem is to find the nonlocal and nonclassical symmetries of FBE. Some general aspects were already presented in [1]. Promising seems an attempt of computer symmetry analysis, as was done for nonlinear heat equation in [4].

Note here that the existence of the transformation T expressed by relation (3.1) allows to solve the problem of symmetry group of FBE (1.6). Let G be a group of symmetry of the diffusion equation, then $G = TG_1T^{-1}$ is a symmetry group of FBE (1.6).

5. CONCLUSIONS AND DISCUSSION

From our point of view, the FBE (1.6), in contrast to the other two possible generalizations of BE, has at least two important advantages:

i) the effect of nonlinearity and nonlocality is concentrated in one term;
ii) the relation with the usual diffusion equation allows a lot of analytical solutions of FBE.

Now we shall highlight the following important aspects. Despite the nonlocality in the proposed nonlinear and nonlocal FBE,

- there are space-localized solutions;
- nonlocal perturbations of such a system can interact;
- the Reynolds number is a universal dimensionless parameter both for the local and nonlocal, fractional Burgers equation.

In some fields of physics we need the vectorial form of the Burgers equation, e.g. in astrophysics to describe the large-scale structure of the Universe [13; 18]. In such cases the vectorial FBE can be proposed:

$$\boldsymbol{\phi}_t + \frac{1}{2}\, {}_a\boldsymbol{D}_{\boldsymbol{x}}^{\boldsymbol{p}} \left({}_a\boldsymbol{D}_{\boldsymbol{x}}^{\boldsymbol{1}-\boldsymbol{p}} \boldsymbol{\phi} \right)^2 - \alpha \boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi}) = 0\,, \tag{5.1}$$

where $\boldsymbol{\phi} = (\phi^{(1)}, \ldots, \phi^{(n)})$, $\boldsymbol{\nabla}$ is the gradient operator and

$${}_a\boldsymbol{D}_{\boldsymbol{x}}^{\boldsymbol{p}} = \boldsymbol{e}_k\, {}_aD_{x_k}^{p_k} = \sum_{k=1}^{n} \frac{\boldsymbol{e}_k}{\Gamma(1-p_k)} \frac{\partial}{\partial x_k} \int_{a_k}^{x_k} \frac{dt}{(x_k - t)^{p_k}}\,, \tag{5.2}$$

with $dt \equiv dt_1 \ldots dt_n$ and $\boldsymbol{e}_k$ are unit normal vector in the x_k direction.

The obtained results, because of their general character, allow a wide range of applications. The process of nonlinear heat distribution in the environment in the presence of heat sources and sinks is a good example.

Our results also apply to the Kardar–Parisi–Zhang (KPZ) equation in the crystal growth phenomena in (1+1)-dimensions [11], the nonlinear dynamics of moving line [10], galaxy formations [13; 20; 21], behavior of magnetic flux line in superconductor [9], and spin glasses [5], as well as to numerous examples of the application of the usual Burgers equation presented in [7].

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Kai kurios trupmeminės Biurgerso lygties savybės

P. Miškinis

Pasiublytas trupmeninis vienmatės Biurgerso lygties apibendrinimas

$$\phi_t + \frac{1}{2}{}_aD_x^p\left({}_aD_x^{1-p}\phi\right)^2 - \alpha\phi_{xx} = 0\,,$$

su pradinėmis sąlygomis $\phi(x,0) = \phi_o(x)$; $\phi_t(x,0) = \psi_o(x)$, kur $\phi = \phi(x,t) \in C^2(\Omega)$; $\phi_t \equiv \partial\phi/\partial t$; ${}_aD_x^p$ yra Rymano bei Liuvilio trupmeninė p eilės išvestinė; $\Omega = (x,t)\colon x \in E^1, t > 0$; bei šios lygties atskiras analitinis sprendinys. Nagrinėjamas impulso bei energijos tvermės dėsnių atitinkami apibendrinimai, sąryšis su paprasta Biurgerso lygtimi ir trupmeninės Hopfo bei Koulo transformacijos savybės.