Mathematical Modelling and Analysis, vol. 3, pp. 104–113
R.Čiegis (Ed)
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NUMERICAL SOLUTIONS AND THEIR SUPERCONVERGENCE FOR WEAKLY SINGULAR INTEGRAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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ABSTRACT

The piecewise polynomial collocation method is discussed to solve second kind Fredholm integral equations with weakly singular kernels $K(t,s)$ which may be discontinuous at $s = d$, $d =$ const. The main result is given in Theorem 4.1. Using special collocation points, error estimates at the collocation points are derived showing a more rapid convergence than the global uniform convergence in the interval of integration available by piecewise polynomials.

1. INTEGRAL EQUATION

Consider the linear integral equation

$$u(t) = \int_0^b K(t,s)u(s)ds + f(t), \quad 0 \le t \le b, \tag{1.1}$$

where $b \in \mathbb{R}$ and $f\colon [0,b] \to \mathbb{R}$ is a given continuous function. Throughout this paper we shall suppose that the kernel K has the form

$$K(t,s) = a(t,s)\kappa(t-s) \tag{1.2}$$

where

(A1) the function $\kappa(\tau)$ is $m-1$ times $(m \ge 1)$ continuously differentiable with respect to τ for $\tau \in [-b,b] \setminus \{0\}$ and such that the estimates

$$|\kappa^{(k)}(\tau)| \le b_k|\tau|^{-\alpha-k}, \quad k = 0,1,\dots,m-1, \tag{1.3}$$

hold with $0 < \alpha < 1$ and some positive constants $b_0, b_1, \ldots, b_{m-1}$;

(A2) the function $a(t,s)$ is m times continuously differentiable on $[0,b] \times [0,d]$ and $[0,b] \times [d,b]$ independently, where d is a fixed point in the interval $(0,b)$.

Let $C^m(X)$, where $X \subset \mathbb{R}$, denote the space of m times continuously differentiable functions $x\colon X \to \mathbb{R}$. For $0 < \alpha < 1$, $m \in \mathbb{N}$, $0 < d < b$, define

$$E^{\alpha,m} \equiv \Big\{ u \in C[0,b] \cap C^m(0,d) \cap C^m(d,b) :$$

$$\sup_{\substack{0<t<b \\ t\neq d}} \frac{|u^{(m)}(t)|}{t^{-(\alpha+m-1)} + |t-d|^{-(\alpha+m-1)} + (b-t)^{-(\alpha+m-1)}} < \infty \Big\};$$

$E^{\alpha,m}$ is a Banach space under the norm

$$||u||_{E^{\alpha,m}} = \max_{0\le t\le b} |u(t)| + \sup_{\substack{0<t<b \\ t\neq d}} \frac{|u^{(m)}(t)|}{t^{-(\alpha+m-1)} + |t-d|^{-(\alpha+m-1)} + (b-t)^{-(\alpha+m-1)}}.$$

It follows from $u \in E^{\alpha,m}$ that $u \in C[0,b] \cap C^m(0,d) \cap C^m(d,b)$ and for $0 < t < d$ and $d < t < b$ the following estimates hold:

$$|u^{(k)}(t)| \le c_k \big[t^{-(\alpha+k-1)} + |t-d|^{-(\alpha+k-1)} + (b-t)^{-(\alpha+k-1)} \big], \ k = 1, \ldots, m, \tag{1.4}$$

where $c_1, \ldots, c_k$ are some positive constants. Note also that $C^m[0,b] \subset E^{\alpha,m}$.

The following result (see [4; 6; 2]) states the regularity properties of solutions of equation (1.1).

LEMMA 1.1. *Let the assumptions* (A1) *and* (A2) *hold, and let* $f \in E^{\alpha,m}$. *If integral equation (1.1) has a solution* $u \in L^1(0,b)$ *then* $u \in E^{\alpha,m}$.

REMARK 1.If the function $a(t,s)$ is continuous on $[0,b] \times [0,b]$ then the estimates (1.4) for the derivatives of the solution $u(t)$ of equation (1.1) can be specified (see [4]).

2. PIECEWISE POLYNOMIAL APPROXIMATION

Let $N \in \mathbb{N}$, $r \in \mathbb{R}$, $r \ge 1$. We introduce in the interval $[0,d]$ the following $2N$ grid points

$$t_j^{(N)} = \left(\frac{j}{N}\right)^r \frac{d}{2}, \ j = 0, 1, \ldots, N; \tag{2.1}$$

$$t_{N+j}^{(N)} = d - t_{N-j}^{(N)}, \ j = 1, \ldots, N-1,$$

and in the interval $[d,b]$ $2N+1$ grid points

$$\begin{aligned} &t_{2N+j}^{(N)} = d + \left(\tfrac{j}{N}\right)^r \tfrac{b-d}{2},\ j = 0,1,\ldots,N; \\ &t_{3N+j}^{(N)} = b - t_{3N-j}^{(N)},\ j = 1,\ldots,N-1; \quad t_{4N}^{(N)} = b. \end{aligned} \tag{2.2}$$

Here $r \geq 1$ characterizes the degree of the nonuniformity of the grid. If $r = 1$ then the grid points (2.1) and (2.2) are uniformly located in the intervals $[0,d]$ and $[d,b]$ respectively (and in $[0,b]$ if $d = b/2$). If $r > 1$ then the grid points $\{(2.1),(2.2)\}$ are more densely located towards the end points of the intervals $[0,d]$ and $[d,b]$.

We dermine the collocation points in the following way. We choose m points $\eta_1,\ldots,\eta_m$ in the interval $[-1,1]$:

$$-1 \leq \eta_1 < \eta_2 < \ldots < \eta_m \leq 1. \tag{2.3}$$

By affine transformations we transfer them into the interval $[t_{j-1}^{(N)}, t_j^{(N)}]$:

$$\xi_{j,q}^{(N)} = t_{j-1}^{(N)} + \frac{\eta_q + 1}{2}(t_j^{(N)} - t_{j-1}^{(N)}), \quad q = 1,\ldots,m;\ j = 1,\ldots,4N. \tag{2.4}$$

Note that $\xi_{j,m}^{(N)} = \xi_{j+1,1}^{(N)} = t_j^{(N)}$, if $\eta_1 = -1$, $\eta_m = 1$ $(j = 1,\ldots,4N-1)$.

For a continuous function $u\colon [0,b] \to \mathbb{R}$ we construct a piecewise polynomial interpolation function $P_N u\colon [0,b] \to \mathbb{R}$ as follows: on every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ $(j = 1,\ldots,4N)$, $P_N u$ is a polynomial of degree not exceeding $m-1$ and

$$(P_N u)(\xi_{j,q}^{(N)}) = u(\xi_{j,q}^{(N)}), \quad q = 1,\ldots,m;\ j = 1,\ldots,4N.$$

Thus the interpolation function $(P_N u)(t)$ is uniquely defined in every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ $(j = 1,\ldots,4N)$ separately and may have jumps if $t = t_j^{(N)}$, $j = 1,\ldots,4N-1$. If $\eta_1 = -1$, $\eta_m = 1$, then $P_N u$ is a continuous function on the interval $[0,b]$. We can define $(P_N u)(t)$ by the formula

$$(P_N u)(t) = \sum_{q=1}^{m} u(\xi_{j,q}^{(N)})\varphi_{j,q}^{(N)}(t), \quad t \in [t_{j-1}^{(N)}, t_j^{(N)}],\ j = 1,\ldots,4N, \tag{2.5}$$

where $\varphi_{j,q}^{(N)}(t)$, $t \in [t_{j-1}^{(N)}, t_j^{(N)}]$, $q = 1,\ldots,m$, are the polynomials of degree $m-1$ such that

$$\varphi_{j,q}^{(N)}(\xi_{j,p}^{(N)}) = \left\{ \begin{array}{ll} 1, & p = q \\ 0, & p \neq q \end{array} \right\}, \quad p = 1,\ldots,m. \tag{2.6}$$

Let us denote by E_N the range of the operator $P_N \equiv P_N^{(m)}$. This is the space of piecewise polynomial functions u_N on $[0,b]$ which on every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ $(j = 1,\ldots,4N)$ are polynomials of the degree not exceeding $m-1$.

The approximation properties of $P_N u$ on grid $\{(2.1),(2.2)\}$ are considered in [5] (cf. also [6; 7; 8]). These results can be summarized as follows.

LEMMA 2.1. *Assume that* $u \in E^{\alpha,m}$. *Then*

$$||u - P_N u||_{L^\infty(0,b)} \leq \text{const} \begin{cases} h_N^{r(1-\alpha)} & \text{for } 1 \leq r \leq \frac{m}{1-\alpha}, \\ h_N^m & \text{for } r \geq \frac{m}{1-\alpha}, \end{cases} \tag{2.7}$$

where

$$h_N = \max\left\{\frac{d}{2N}, \frac{b-d}{2N}\right\}. \tag{2.8}$$

3. COLLOCATION METHOD

We look for an approximate solution $u_N \in E_N$ to integral equation (1.1). We require that u_N should satisfy the equation (1.1) at the collocation points (2.4):

$$\left[u_N(t) - \int_0^b K(t,s)u_N(s)ds - f(t)\right]_{t=\xi_{i,p}^{(N)}} = 0, \quad p = 1,\ldots,m, \quad i = 1,\ldots,4N. \tag{3.1}$$

By the representation (2.5), we can find $u_N \in E_N$ in the form

$$u_N(t) = \sum_{q=1}^m c_{j,q}^{(N)} \varphi_{j,q}^{(N)}(t), \quad t \in [t_{j-1}^{(N)}, t_j^{(N)}],\ j = 1\ldots,4N,$$

where, as it follows from (2.6),

$$c_{j,q}^{(N)} = u_N(\xi_{j,q}^{(N)}), \quad q = 1,\ldots,m;\ j = 1,\ldots,4N.$$

Now the collocation conditions (3.1) will take the following form of a system which determines the coefficients $c_{i,p}^{(N)} = u_N(\xi_{i,p}^{(N)})$:

$$c_{i,p}^{(N)} = \sum_{j=1}^{4N}\sum_{q=1}^m a_{i,p,j,q}^{(N)} c_{j,q}^{(N)} + f(\xi_{i,p}^{(N)}), \quad p = 1,\ldots,m;\ i = 1,\ldots,4N, \tag{3.2}$$

where

$$a_{i,p,j,q}^{(N)} = \int_0^b K(\xi_{i,p}^{(N)}, s)\varphi_{j,q}^{(N)}(s)ds.$$

If $\eta_1 > -1$ or $\eta_m < 1$, then all collocation points $\xi_{j,q}^{(N)}$ ($q = 1,\ldots,m$, $j = 1,\ldots 4N$) are different and there are $4mN$ collocation points. In this case the system (3.2) (system (3.1)) has $4mN = \dim E_N$ equations and the same number of unknows. If $\eta_1 = -1$ and $\eta_m = 1$, then part of the collocation points will coincide. The number of different collocation points is $[4N(m-1)+1] = \dim E_N$ and the system (3.2) (system (3.1)) has the same number of equations and unknows.

THEOREM 3.1. (cf. [5]). *Assume that the following conditions are fulfilled:* 1) *the kernel* (1.2) *satisfies the assumptions* (A1) *it and* (A2)*;* 2) $f \in E^{\alpha,m}$*;* 3) *the homogeneous integral equation*

$$u(t) = \int_0^b K(t,s)u(s)ds \tag{3.3}$$

has only the trivial solution $u = 0$*;* 4) *the collocation points* (2.4) *are used. Then the equation* (1.1) *has a unique solution* u^* *and there exists* N_0 *such that, for* $N \geq N_0$*, the collocation conditions* (3.1) *define a unique approximation* $u_N^* \in E_N$ *to* u^**. The following error estimates hold:*

$$||u_N^* - u^*||_{L^\infty(0,b)} \leq c \begin{cases} h_N^{r(1-\alpha)} & \text{for } 1 \leq r \leq \frac{m}{1-\alpha}, \\ h_N^m & \text{for } r \geq \frac{m}{1-\alpha}, \end{cases} \tag{3.4}$$

where r *is the scaling parameter of the grid* $\{(2.1),(2.2)\}$*,* h_N *is defined in* (2.8) *and* c *is a positive constant independent of* h_N*.*

Proof. We write the integral equation (1.1) in the form $u = Tu + f$ where

$$(Tu)(t) = \int_0^b K(t,s)u(s)ds, \quad t \in [0,b]. \tag{3.5}$$

It follows from (A1) and (A2) (see [1]) that $T: L^\infty(0,b) \to C[0,b]$, moreover, $T: L^\infty(0,b) \to L^\infty(0,b)$, is compact. As the homogeneous equation $u = Tu$ has only the trivial solution $u = 0$, then the equation $u = Tu + f$ has a unique solution $u^* \in L^\infty(0,b)$. Due to Lemma 1.1, $u^* \in E^{\alpha,m}$. The collocation conditions (3.1) can be written in the form

$$u_N = P_N T u_N + P_N f \tag{3.6}$$

where P_N is defined in Section 2. If $N \to \infty$ then $||P_N u - u||_{L^\infty(0,b)} \to 0$ for every $u \in C[0,b]$. Therefore $||P_N T - T||_{L^\infty(0,b) \to L^\infty(0,b)} \to 0$, $N \to \infty$. From this and from the boundedness of $(I-T)^{-1}$ in $L^\infty(0,b)$ we obtain that $I - P_N T$ is invertible for sufficiently large $N \geq N_0$ and uniformly bounded in N:

$$||(I - P_N T)^{-1}||_{L^\infty(0,b) \to L^\infty(0,b)} \leq c. \tag{3.7}$$

Let $N \geq N_0$ and $u_N^* = (I - P_N T)^{-1} P_N f$ be the solution of the equation (3.6). Then $u_N^* - u^* = (I - P_N T)^{-1}(P_N u^* - u^*)$ and

$$||u_N^* - u^*||_{L^\infty(0,b)} \leq c||P_N u^* - u^*||_{L^\infty(0,b)}.$$

Due to (2.7) we obtain the estimate (3.4). □

4. SUPERCONVERGENCE AT COLLOCATION POINTS

Now we assume that the points (2.3) are the nodes of a quadrature formula

$$\int_{-1}^{1} g(s)ds = \sum_{q=1}^{m} A_q g(\eta_q) + R_m(g), \quad -1 \leq \eta_1 < \ldots < \eta_m \leq 1, \tag{4.1}$$

which is exact for all polynomials of degree $m+1$ ($m \geq 2$).

THEOREM 4.1. *Let $m \in \mathbb{N}$, $m \geq 2$. Assume that the following conditions are fulfilled:*

(i) the kernel K has the form (1.2) *where*

1) the function $\kappa(\tau)$ is m times continuously differentiable with respect to τ for $\tau \in [-b, b] \setminus \{0\}$ and such that the estimates

$$|\kappa^{(k)}(\tau)| \leq b_k |\tau|^{-\alpha-k}, \quad k = 0, 1, \ldots, m,$$

hold with $0 < \alpha < 1$ and some positive constants $b_0, b_1, \ldots, b_m$;

2) the function $a(t,s)$ is $m+1$ times continuously differentiable on $[0,b] \times [0,d]$ and $[0,b] \times [d,b]$, where $0 < d < b$;

(ii) $f \in E^{\alpha,m+1}$;

(iii) homogeneous integral equation (3.3) *has only the trivial solution $u = 0$;*

(iv) the grid $\{(2.1),(2.2)\}$ is used with $r \geq (m+1)/(1-\alpha)$ and the collocation points (2.4) *are generated by the nodes* (2.3) *of a quadrature formula* (4.1) *which is exact for all polynomials of degree $m+1$.*

Then there exists $N_0 \in \mathbb{N}$ such that for $N \geq N_0$

$$\max_{q=1,\ldots,m;j=1,\ldots,4N} |u_N^*(\xi_{j,q}^{(N)}) - u^*(\xi_{j,q}^{(N)})| \leq c h_N^m h_N^{1-\alpha}, \tag{4.2}$$

where u^* is the solution of equation (1.1), *$u_N^* \in E_N$ is the solution of the system* (3.1), *h_N is defined in* (2.8) *and c is a positive constant independent of h_N (of N).*

Proof. Due to Lemma 1.1, $u^* \in E^{\alpha,m+1}$. We have

$$|u_N^*(\xi_{j,q}^{(N)}) - u^*(\xi_{j,q}^{(N)})| \le ||u_N^* - P_N u^*||_{L^\infty(0,b)} \quad (q = 1, \dots, m; j = 1, \dots, 4N). \tag{4.3}$$

As

$$u_N^* - P_N u^* = (I - P_N T)^{-1} P_N T (P_N u^* - u^*) \quad N \ge N_0,$$

then with help of (3.7)

$$||u_N^* - P_N u^*||_{L^\infty(0,b)} \le c \, ||T(P_N u^* - u^*)||_{L^\infty(0,b)}. \tag{4.4}$$

Let us estimate $||T(P_N u^* - u^*)||_{L^\infty(0,b)}$. Fix $t \in [0,b]$ and let

$$\eta(t, h_N) = (t - h_N, t + h_N) \cap [0,b]. \tag{4.5}$$

Then

$$\left| \int_0^b K(t,s)[u^*(s) - (P_N u^*)(s)] ds \right| \le I_1(t) + I_2(t), \tag{4.6}$$

where

$$I_1(t) = \sum_{j:[t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t,h_N) \ne 0} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |K(t,s)[u^*(s) - (P_N u^*)(s)]| ds,$$

$$I_2(t) = \sum_{j:[t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t,h_N) = 0} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |K(t,s)[u^*(s) - (P_N u^*)(s)]| ds.$$

It follows from the assumption (i) that

$$I_1(t) \le c ||u^* - P_N u^*||_{L^\infty(0,b)} \sum_{j:[t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t,h_N) \ne 0} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |t - s|^{-\alpha} ds.$$

By the Lemma 2.1 we obtain $||u^* - P_N u^*||_{L^\infty(0,b)} \le c' h_N^m$. Due to (4.5)

$$\sum_{j:[t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t,h_N) \ne 0} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |t - s|^{-\alpha} ds \le c'' \int_{t-2h_N}^{t+2h_N} |t - s|^{-\alpha} ds \le c''' h_N^{m+1-\alpha}.$$

Thus

$$I_1(t) \le c_1 h_N^m h_N^{1-\alpha}, \quad t \in [0,b], \quad c_1 = \text{const}. \tag{4.7}$$

Consider the term $I_2(t)$, $t \in [0,b]$. Let

$$t_{j,\frac{1}{2}}^{(N)} = \frac{t_{j-1}^{(N)} + t_j^{(N)}}{2}.$$

In addition to the points (2.3) we fix in $[-1,1]$ a point η_{m+1} ($\eta_{m+1} \ne \eta_i$, $i = 1,\dots,m$). By an affine transformation we transfer η_{m+1} into the point $\xi_{j,m+1}^{(N)} \in [t_{j-1}^{(N)}, t_j^{(N)}]$ so that $\xi_{j,m+1}^{(N)} \ne \xi_{j,i}^{(N)}$, $i = 1,\dots,m$ ($j = 1,\dots 4N$). Similarly to the definition of P_N (see Section 2) we define for a continuous function $u\colon[0,b] \to \mathbb{R}$ a piecewise polynomial function $P_N^{(m+1)}u\colon[0,b] \to \mathbb{R}$ as follows: $P_N^{m+1}u$ is on every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ ($j = 1,\dots,4N$) a polynomial of degree not exceeding m and

$$P_N^{(m+1)}u(\xi_{j,q}^{(N)}) = u(\xi_{j,q}^{(N)}), \quad q = 1,\dots,m+1; j = 1,\dots,4N.$$

We have

$$I_2(t) \le I_{21}(t) + I_{22}(t) + I_{23}(t), \quad t \in [0,b],$$

where

$$I_{21}(t) = \sum_{j:[t_{j-1}^{(N)},t_j^{(N)}]\cap\eta(t,h_N)=\emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} \left|K(t,s) - K(t,t_{j,\frac{1}{2}}^{(N)})\right| |u^*(s) - (P_N u^*)(s)| ds,$$

$$I_{22}(t) = \sum_{j:[t_{j-1}^{(N)},t_j^{(N)}]\cap\eta(t,h_N)=\emptyset} \left|K(t,t_{j,\frac{1}{2}}^{(N)})\right| \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |u^*(s) - (P_N u^*)(s)| ds,$$

$$I_{23}(t) = \left| \sum_{j:[t_{j-1}^{(N)},t_j^{(N)}]\cap\eta(t,h_N)=\emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} K(t,t_{j,\frac{1}{2}}^{(N)})[(P_N^{(m+1)}u^*)(s) - (P_N u^*)(s)] ds \right|.$$

Let us consider $I_{21}(t)$, $t \in [0,b]$. It follows from Lemma 2.1 that

$$I_{21}(t) \le c h_N^m \sum_{j:[t_{j-1}^{(N)},t_j^{(N)}]\cap\eta(t,h_N)=\emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} \left|\frac{\partial K(t,s)}{\partial s}\right|_{s=\tau_j} \left|s - t_{j,\frac{1}{2}}^{(N)}\right| ds,$$

where $\tau_j \in (s, t_{j,\frac{1}{2}}^{(N)})$. We have for $s \in [t_{j-1}^{(N)}, t_j^{(N)}]$

$$\left|\frac{\partial K(t,s)}{\partial s}\right|_{s=\tau_j} |s - t_{j,\frac{1}{2}}^{(N)}| \le c' h_N |t - \tau_j|^{-\alpha-1}.$$

Since $[t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) = \emptyset$, $s \in [t_{j-1}^{(N)}, t_j^{(N)}]$ and $\tau_j \in (s, t_{j,\frac{1}{2}}^{(N)})$, then

$$\tilde{c}_1 \leq \frac{|t-\tau_j|}{|t-s|} \leq \tilde{c}_2,$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are some positive constants. Therefore

$$I_{21}(t) \leq c'' h_N^{m+1} \sum_{j:[t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t,h_N) = \emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |t-s|^{-\alpha-1} ds \leq$$

$$\leq c''' h_N^{m+1} \int_{[0,b]\setminus\eta(t,h_N)} |t-s|^{-\alpha-1} ds.$$

Due to (4.5) $\int_{[0,b]\setminus\eta(t,h_N)} |t-s|^{-\alpha-1} ds \leq c'''' h_N^{-\alpha}$. Thus

$$I_{21}(t) \leq c_2 h_N^m h_N^{1-\alpha}, \quad t \in [0,b],\ c_2 = \text{const.}$$

Let us turn to $I_{22}(t)$, $t \in [0,b]$. It follows from $[t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) = \emptyset$ that $|K(t, t_{j,\frac{1}{2}}^{(N)})| \leq c|t - t_{j,\frac{1}{2}}^{(N)}|^{-\alpha} \leq c' h_N^{-\alpha}$. Due to Lemma 2.1 $||u^* - P_N^{(m+1)} u^*||_{L^\infty(0,b)} \leq c'' h_N^{m+1}$. Therefore

$$I_{22}(t) \leq c_3 h_N^m h_N^{1-\alpha}, \quad t \in [0,b],\ c_3 = \text{const.}$$

Consider $I_{23}(t)$, $t \in [0,b]$. Due to the assumption (iv) we obtain that the quadrature formula

$$\int_{t_{j-1}^{(N)}}^{t_j^{(N)}} g(s) ds = \frac{t_j^{(N)} - t_{j-1}^{(N)}}{2} \sum_{q=1}^{m} A_q g(\xi_{j,q}^{(N)}) + \frac{t_j^{(N)} - t_{j-1}^{(N)}}{2} R_m(g)$$

remains to be exact for polynomials of degree $m+1$. Using this we have

$$\int_{t_{j-1}}^{t_j} [(P_N^{(m+1)} u^*)(s) - (P_N u^*)(s)] ds = 0,$$

and therefore $I_{23}(t) = 0$, $t \in [0,b]$. Thus

$$I_2(t) \leq c_4 h_N^m h_N^{1-\alpha}, \quad t \in [0,b],\ c_4 = \text{const.} \tag{4.8}$$

Now the estimate (4.2) follows from (4.3), (4.4), (4.6), (4.7) and (4.8). □

REMARK 2. For $a \in C^{m+1}([0,b] \times [0,b])$ the estimate (4.2) follows from the corresponding results in [6; 3].

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