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SOME HERONIAN MEAN OPERATORS WITH 2-TUPLE LINGUISTIC INFORMATION AND THEIR APPLICATION TO MULTIPLE ATTRIBUTE GROUP DECISION MAKING

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Abstract. With respect to multi-attribute group decision-making problems, in which attribute values take the form of 2-tuple linguistic information, a new decision making method that considers the interrelationships of attribute values is proposed. Firstly, some new aggregation operators of 2-tuple linguistic information based on Heronian mean are proposed, such as 2-tuple linguistic Heronian mean operator (2TLHM) and 2-tuple linguistic weighted Heronian mean operator (2TLWHB), and some desired properties of the proposed operators are studied. Then, a method based on the 2TLHM and 2TLWHB operators for multiple attribute group decision making is developed. In this approach, the interrelationships of attribute values are considered. Finally, an illustrative example is given to verify the developed approach and to demonstrate its practicality and effectiveness.

Keywords: 2-tuple linguistic information, Heronian mean operator, 2-tuple linguistic Heronian mean operator (2TLHM), 2-tuple linguistic weighted Heronian mean operator (2TLWHB), multiple attribute group decision making (MAGDM).

JEL Classification: C44, C60.

Introduction

Since most of the decision problems are fuzzy and uncertain, sometimes, the attributes involved in these decision problems are difficult to be expressed by crisp numbers (Filip *et al.* 2014; Su *et al.* 2013; Yu 2013; Liu 2013, 2014a, 2014b; Liu, Jin 2012; Liu, P. D., Liu, Y. 2014; Liu, Wang 2014), especially for some qualitative information, they can be described by the linguistic terms directly, such as "good", "better" or "poor". Since Zadeh (1975a, 1975b, 1975c) proposed the concept of linguistic variables, the researches on linguistic multiple at-

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tribute decision making problems have got a lot of achievements, especially, the aggregation operators for the linguistic variables have been proposed to solve the linguistic multiple attribute decision making problems. Herrera *et al.* (1995, 1996b) proposed the linguistic ordered weighted averaging (LOWA) operators to aggregate linguistic preference relations. Xu (2004a) proposed the linguistic order weighted geometric (LOWG) operators. Wei (2011) proposed the 2-tuple linguistic weighted harmonic averaging (TWHA) operator, 2-tuple linguistic ordered weighted harmonic averaging (TOWHA) operator and 2-tuple linguistic combined weighted harmonic averaging (TCWHA) operator, and analyzed their properties. Further, a method based on the TWHA and TCWHA operators for the MAGDM problems is developed. Xu (2004b) studied the group decision making problems, in which all weights of the attributes and the decision-makers, and attribute values take the form of linguistic terms. Then, the author defined the operational laws of the linguistic evaluation, developed some new operators, and proposed a method based on the operators for MAGDM under pure linguistic information. Zhang (2012) proposed some aggregation operators of interval-valued 2-tuples, such as interval-valued 2-tuple weighted average (IVTWA) operator, interval-valued 2-tuple ordered weighted average (IVTOWA) operator. Further, their properties are studied, and the MAGDM method based on these operators with interval-valued 2-tuple linguistic information is proposed. Merigó *et al.* (2010) proposed linguistic ordered weighted averaging operator and the linguistic hybrid averaging operator based on Dempster-Shafer theory of evidence. Merigó and Gil-Lafuente (2013) proposed the induced 2-tuple linguistic generalized aggregation operators, and applied them to solve the multiple attribute decision-making problems. Xu *et al.* (2013) proposed some proportional 2-tuple geometric aggregation operators for linguistic decision making.

The common characteristics of the above aggregation operators are that they emphasize the importance of each datum or their ordered position, but cannot reflect the interrelationships of the individual data. In order to deal with the interrelationships of attribute values, Xu and Wang (2011) proposed some new linguistic aggregation operators, such as 2-tuple linguistic power average (2TLPA) operator, 2-tuple linguistic weighted PA operator (2TLWPA), and 2-tuple linguistic power ordered weighted PA operator (2TLPOWA) operator, in order to take all the decision arguments and their relationships into account, and studied some desired properties of the developed operators, such as idempotency and boundedness. Further, two approaches to deal with group decision making problems under linguistic environment are proposed. Yang and Chen (2012) proposed some new aggregation operators – including the 2-tuple correlated averaging operator, the 2-tuple correlated geometric operator and the generalized 2-tuple correlated averaging operator – to deal with the group decision making problem with inter-dependent or interactive attributes. In addition, they proposed a new multiple attribute decision making method based on the new operators. Wei and Zhao (2012) proposed some dependent 2-tuple linguistic aggregation operators, such as the dependent 2-tuple ordered weighted averaging (D2TOWA) operator and the dependent 2-tuple ordered weighted geometric (D2TOWG) operator in which the associated weights only depend on the aggregated 2-tuple linguistic arguments. These operators can relieve the influence of unfair 2-tuple linguistic arguments on the aggregated results by assigning low weights to those "false" and "biased" ones. Then, some approaches for MAGDM with 2-tuples linguistic information were proposed.

Heronian mean is a very important operator which can also consider the data interrelationships. However, in the past, it was applied to the theories and applications of inequality. In recent years, Beliakov *et al.* (2007) have firstly proved that it was an aggregation operator. Sykora (2009b) further proposed the generalized Heronian method. Sykora (2009a) analyzed two special cases of generalized Heronian methods. Liu and Pei (2012) proposed Heronian mean operator and Heronian OWA operator with the parameters respectively, which were similar to Bonferroni mean operator and BON-OWA operator proposed by Bonferroni (1950) and Yager (2009), and compared with the previous methods. It was shown that this operator has the advantage of considering the interrelationships between the attributes. Yu and Wu (2012) compared Heronian mean operator with PA operator, Bonferroni mean operator and Choquet integral operator, and further extended Heronian mean operator to process intuitionistic fuzzy numbers. Liu *et al.* (2014) proposed intuitionistic uncertain linguistic arithmetic Heronian mean operator, intuitionistic uncertain linguistic weighted arithmetic Heronian mean operator, intuitionistic uncertain linguistic geometric Heronian mean operator, and intuitionistic uncertain linguistic weighted geometric Heronian mean operator, and applied them to multiple attribute group decision making. Chen and Liu (2014) further extended Heronian mean to intuitionistic trapezoidal fuzzy numbers, and proposed intuitionistic trapezoidal fuzzy general Heronian OWA operator.

In the real decision making problems, there exists the interactions between the attributes. Because Heronian mean operator can process the interactions between the attributes and linguistic variables are easy to express the fuzzy information, it is important to extend Heronian mean operator to deal with linguistic information. Therefore, this paper is aimed at multi-attribute decision-making problems in which attribute values are linguistic variables, by combining the Heronian mean operator with linguistic variables expressed in 2-tuple, we will propose some 2-tuple generalized Heronian mean operators, then applies them to multi-attribute decision-making problems. To do this, the structure of this paper is arranged shown as follows. In section 1, we briefly review some basic concepts of 2-tuple, Bonferroni mean operator and Heronian mean operator. In section 2, we propose the 2-tuple linguistic Heronian mean operator and 2-tuple linguistic weighted Heronian mean operator, and discuss some desirable properties of these operators. In section 3, we develop a method for MAGDM problems based on the proposed operators. Section 4 gives an example to illustrate the decision steps and discusses the influence of different parameters in these operators on the decision-making results, and compares with existing method. In the last section, we give the conclusions and future research directions.

1. Preliminaries

1.1. 2-tuple linguistic information

Suppose that $S=(s_0,s_1,\cdots,s_{l-1})$ is a finite and fully ordered discrete term set, where l is an odd number. In real situations, l would be equal to 3, 5, 7, 9, etc. For example, when l = 7, a set S can be given as follows:

$S=(s_0,s_1,s_2,s_3,s_4,s_5,s_6)$ = {very poor, poor, slightly poor, fair, slightly good, good, very good}.

Usually, for any linguistic set S, it should satisfy the following characteristics (Herrera *et al.* 1996a; Herrera, Herrera-Viedma 2000):

(1) The set is ordered: $s_i < s_j$, if and only if $i < j$; (2) There is the negation operator: $neg(s_i) = s_{l-i-1}$; (3) Maximum operator: $\max(s_i, s_j) = s_i$, if $i \geq j$; (4) Minimum operator: $\min(s_i, s_j) = s_i$, if $i \leq j$.

For any linguistic set $S = (s_0, s_1, \cdots, s_{l-1})$, the relationship between the element s_i and its subscript i is strictly monotonically increasing (Herrera *et al.* 1996a; Xu 2006a), so the function can be defined as follows: $f : s_i = f(i)$. Clearly, the function $f(i)$ is a strictly monotonically increasing function about a subscript i. To preserve all of the given information, the discrete linguistic label $S = (s_0, s_1, \cdots, s_{l-1})$ is extended to a continuous linguistic label $\overline{S} = \{s_\alpha \mid \alpha \in R\}$, which satisfies the above characteristics.

The operational laws for the linguistic label are defined as follows (Xu 2006b):

$$\beta s_i = s_{\beta \times i} \quad \beta \geq 0; \tag{1}$$

$$s_i \oplus s_j = s_{i+j}; \tag{2}$$

$$s_i / s_j = s_{i/j}, j \neq 0; \tag{3}$$

$$\left(s_i\right)^n = s_{i^n}; \tag{4}$$

$$\lambda(s_i \oplus s_j) = \lambda s_i \oplus \lambda s_j \quad \lambda \geq 0; \tag{5}$$

$$(\lambda_1 + \lambda_2) s_i = \lambda_1 s_i \oplus \lambda_2 s_i \quad \lambda_1, \lambda_2 \geq 0; \tag{6}$$

$$\left(s_i\right)^{\lambda_1} \otimes \left(s_i\right)^{\lambda_2} = \left(s_i\right)^{\lambda_1 + \lambda_2} \lambda_1, \lambda_2 \geq 0. \tag{7}$$

In order to process the linguistic information easily, Herrera and Martinez (2000) proposed a symbolic translation method by concept of 2-tuple. In the following, we can give the relevant definitions.

Definition 1. (Herrera, Martinez 2000; Herrera *et al.* 2005): Let $S = (s_0, s_1, \cdots, s_{l-1})$ be a linguistic term set, β is a real number in $[0, l-1]$, which represents the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β can be obtained from the following function:

$$\Delta : [0, l-1] \to S \times [-0.5, 0.5);$$

$$\Delta(\beta) = (s_i, \alpha), \tag{8}$$

where $i = \text{round}(\beta)$, $\alpha = \beta - i$, $\alpha \in [-0.5, 0.5)$, and $\text{round}(.)$ is the usual round operation.

Definition 2. (Herrera, Martinez 2000; Herrera *et al.* 2005): Let $S = (s_0, s_1, \cdots, s_{l-1})$ be a linguistic term set, and (s_i, α) be a 2-tuple, then there is an inverse function Δ^{-1} which can convert a 2-tuple to the corresponding real number $\beta \in [0, l-1]$, that is:

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [0, l-1];$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta. \tag{9}$$

Based on the above definitions, we can know that the 2-tuple corresponding to the element $s_i (i = 0, 1, \cdots, l-1)$ is $(s_i, 0)$.

In addition, we can also give the related calculation models of 2-tuple shown as follows.

(1) There is an inverse operator Neg (Herrera, Martinez 2000):

$\mathrm{Neg}(s_i,\alpha)=\Delta((l-1)-(\Delta^{-1}(s_i,\alpha)))$.

(2) The comparison of 2-tuple: Let (s_i, α_1) and (s_j, α_2) be any two 2-tuple, and there are the following rules for comparison (Herrera, Martinez 2000):

If $i > j$, then $(s_i, \alpha_1) > (s_j, \alpha_2)$, which means (s_i, α_1) is superior to (s_j, α_2);

If $i = j$ then

(a) if $\alpha_1 = \alpha_2$, then $(s_i, \alpha_1) = (s_j, \alpha_2)$, which means (s_i, α_1) is the same as (s_j, α_2);

(b) if $\alpha_1 > \alpha_2$, then $(s_i, \alpha_1) > (s_j, \alpha_2)$, which means (s_i, α_1) is superior to (s_j, α_2);

(c)if $\alpha_1 < \alpha_2$, then $(s_i, \alpha_1) < (s_j, \alpha_2)$, which means (s_i, α_1) is inferior to (s_j, α_2).

(3) If $(s_i, \alpha_1) \geq (s_j, \alpha_2)$, then $\max\{(s_i, \alpha_1), (s_j, \alpha_2)\} = (s_i, \alpha_1)$;
and if $(s_i, \alpha_1) \leq (s_j, \alpha_2)$, then $\min\{(s_i, \alpha_1), (s_j, \alpha_2)\} = (s_i, \alpha_1)$.

1.2. Bonferroni mean operator

The Bonferroni mean (BM) was originally proposed by Bonferroni (1950), which can capture the interrelationship between the individual data. It was defined as follows.

Definition 3. (Bonferroni 1950): Let $I=[0,1]$, $p,q\geq 0$, $B^{p,q}: I^n \to I$ .If $B^{p,q}$ satisfies:

$$B^{p,q}(x_1,x_2,\ldots,x_n)=\left(\frac{1}{n}\frac{1}{n-1}\sum_{i,j=1,i\neq j}^{n} x_i^p x_j^q\right)^{\frac{1}{p+q}}, \tag{10}$$

Then function $B^{p,q}$ is called Bonferroni mean (BM) operator.

Obviously, the BM operator has some desired properties, such as commutativity, idempotency, monotonicity and boundedness, etc**.**

Specifically, when $p = q = 1$, then (10) reduces to the following form:

$$B^{1,1}(a_1,a_2,\cdots,a_n)=\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} a_i\, a_j\right)^{1/2}. \tag{11}$$

Furthermore, we can get

$$B^{1,1}(x_1,x_2,\ldots,x_n)=\left(\frac{1}{n}\sum_{i=1}^{n}u_i x_i\right)^{\frac{1}{2}}, \tag{12}$$

where $u_i=\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n}x_j$.

1.3. Heronian mean (HM) operator

Similar to Bonferroni mean operator, Heronian mean (HM) is also an important aggregation operator, which can also capture the interrelationship of the individual argument (Beliakov *et al.* 2007; Liu, Pei 2012). It can be defined as follows.

Definition 4. (Liu, Pei 2012): Let $I=[0,1]$, $H: I^n \to I$, if H satisfies:

$$H(x_1,x_2,...,x_n)=\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{x_i x_j}. \tag{13}$$

Then function H is called Heronian mean (HM) operator.

Definition 5. (Liu, Pei 2012): Let $I=[0,1]$, $p,q \ge 0$, $H^{p,q}: I^n \to I$, If $H^{p,q}$ satisfies:

$$H^{p,q}(x_1,x_2,...,x_n)=\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}x_i^p x_j^q\right)^{\frac{1}{p+q}}, \tag{14}$$

Then function $H^{p,q}$ is called Heronian mean operator with parameter.

Obviously, the HM operator has the following properties (Liu, Pei 2012).

Theorem 1 (Idempotency):

Let $x_j = x, j=1,2,\cdots,n$, then $H^{p,q}(x,x...,x)=x$.

Theorem 2 (Monotonicity):

Let $(x_1,x_2,\cdots,x_n)$ and $(y_1,y_2,\cdots,y_n)$ be two sets of the real numbers, if $x_j \le y_j$ for all $j=1,2,\cdots,n$, then $H^{p,q}(x_1,x_2...,x_n) \le H^{p,q}(y_1,y_2...,y_n)$.

Theorem 3 (Boundedness):

HM operator lies between the max and min operators, i.e.

$\min\{x_1,x_2...,x_n\} \le H^{p,q}(x_1,x_2...,x_n) \le \max\{x_1,x_2...,x_n\}$.

Some special cases of the $H^{p,q}$ operator are shown as follows.

(1) when $p = q$, then:

$$H^{p,p}(x_1,x_2,...,x_n)=(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}(x_i x_j)^p)^{\frac{1}{2p}}. \tag{15}$$

Further, when $p=q=\frac{1}{2}$, $H^{\frac{1}{2},\frac{1}{2}}(x_1,x_2...,x_n)=H(x_1,x_2...,x_n)$. Therefore, the operator $H^{p,q}$ is a generalization of Heronian mean operator.

(2) when $p = q = 1$, there is

$$H^{1,1}(x_1,x_2,...,x_n)=(\frac{1}{n}\sum_{i=1}^{n}u_i x_i)^{\frac{1}{2}}, \tag{16}$$

where $u_i=\frac{1}{n+1}(x_i+\sum_{j=1}^{n}x_j)$.

1.4. Comparison of BM operator with HM operator

Liu and Pei (2012) had proved the relationship between BM and HM operators as follows.

$$H^{p,p}(x_1,x_2,...,x_n)=\left(\frac{n-1}{n+1}\left[H^{p,p}(x_1,x_2,...,x_n)\right]^{p+q}+\frac{2}{n+1}\left[GOWA(x_1,x_2,...,x_n)\right]^{p+q}\right)^{\frac{1}{p+q}},$$

where GOWA is the generalized ordered weighted aggregation operator (Liu, Pei 2012).

In order to clearly explain the relationship between BM and HM operators, we can compare (12) and (16) with respect to $p = q = 1$.

Formulas (12) and (16) have the same form, but the parameter u_i is different, we use u_i^B to represent u_i in (12) and u_i^H to represent u_i in (16).

$$u_i^B = \frac{1}{n-1}\sum_{\substack{j=1 \\ j\neq i}}^{n} x_j \text{ , } u_i^H = \frac{1}{n+1}(x_i + \sum_{j=1}^{n} x_j) .$$

Obviously, u_i^B and u_i^H are equivalent to the weight of input, u_i^B considers only the average of the input data except of x_i, and u_i^H assigns more importance to x_i when averaging all the input data.

2. 2-tuple linguistic Heronian mean operators

The Heronian mean (HM) operator is an important aggregation operator, however, it has usually been used in situations in which the input arguments are the real numbers. In this section, we shall extend the HM operator to the situations in which the input arguments are linguistic information.

We can give the definition of the 2-tuple linguistic Heronian mean (2*TLHM*) operator as follows.

Definition 6. Let $a = \{(s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\}$ be a collection of 2-tuple, and $2TLHM : \Omega^n \to \Omega$, if

$$2TLHM^{p,q}\left((s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\right) = \Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}(s_i,\alpha_i)\right)^p\left(\Delta^{-1}(s_j,\alpha_j)\right)^q\right)^{1/p+q}\right), \tag{17}$$

where Ω is the set of all 2-tuple, and for any $p,q \geq 0$, then $2TLHM^{p,q}$ is called the 2-tuple linguistic Heronian mean (2TLHM) operator.

The $2TLHM^{p,q}$ operator has the following properties:

(1) Theorem 4 (Idempotency):

Let $a = \{(s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\}$ be a collection of 2-tuple. If $(s_j,\alpha_j) = (s,\alpha)$ for all *j*, then $2TLHM^{p,q}\left((s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\right) = (s,\alpha)$

Proof. Since $(s_j,\alpha_j) = (s,\alpha)$, for all *j*, we have:

$$2TLHM^{p,q}\left((s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\right) = \Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}(s_i,\alpha_i)\right)^p\left(\Delta^{-1}(s_j,\alpha_j)\right)^q\right)^{1/p+q}\right) =$$

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}(s,\alpha)\right)^{p+q}\right)^{1/p+q}\right) =$$

$$\Delta\left(\left(\left(\Delta^{-1}(s,\alpha)\right)^{p+q}\right)^{1/p+q}\right)=\Delta\left(\Delta^{-1}(s,\alpha)\right)=(s,\alpha).$$

(2) Theorem 5 (Monotonicity):

Let $a=\{(s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\}$ and $a'=\{(s'_1,\alpha'_1),(s'_2,\alpha'_2),\cdots,(s'_n,\alpha'_n)\}$ be two collections of 2-tuple. If $(s_j,\alpha_j)\geq(s'_j,\alpha'_j)$ for all *j*, then

$2TLHM^{p,q}\left((s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\right)\geq 2TLHM^{p,q}\left((s'_1,\alpha'_1),(s'_2,\alpha'_2),\cdots,(s'_n,\alpha'_n)\right).$

Proof. Since $(s_j,\alpha_j)\geq(s'_j,\alpha'_j)$ for all *j*, we have:

$$\left(\Delta^{-1}(s_i,\alpha_i)\right)^p\left(\Delta^{-1}(s_j,\alpha_j)\right)^q\geq\left(\Delta^{-1}(s'_i,\alpha'_i)\right)^p\left(\Delta^{-1}(s'_j,\alpha'_j)\right)^q.$$

Further have:

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}(s_i,\alpha_i)\right)^p\left(\Delta^{-1}(s_j,\alpha_j)\right)^q\geq\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}(s'_i,\alpha'_i)\right)^p\left(\Delta^{-1}(s'_j,\alpha'_j)\right)^q$$

and

$$\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}(s_i,\alpha_i)\right)^p\left(\Delta^{-1}(s_j,\alpha_j)\right)^q\right)^{1/p+q}\right)\geq$$

$$\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}(s'_i,\alpha'_i)\right)^p\left(\Delta^{-1}(s'_j,\alpha'_j)\right)^q\right)^{1/p+q}\right).$$

So,

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}(s_i,\alpha_i)\right)^p\left(\Delta^{-1}(s_j,\alpha_j)\right)^q\right)^{1/p+q}\right)\geq$$

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}(s'_i,\alpha'_i)\right)^p\left(\Delta^{-1}(s'_j,\alpha'_j)\right)^q\right)^{1/p+q}\right)$$

i.e., $\left(\Delta^{-1}(s_i,\alpha_i)\right)^p\left(\Delta^{-1}(s_j,\alpha_j)\right)^q\geq\left(\Delta^{-1}(s'_i,\alpha'_i)\right)^p\left(\Delta^{-1}(s'_j,\alpha'_j)\right)^q.$

(3) Theorem 6 (Boundedness):

Let $a=\{(s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\}$ be a collection of 2-tuple, and $(s^-,\alpha^-)=\min\left((s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\right)$, $(s^+,\alpha^+)=\max\left((s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\right)$, then $2TLHM^{p,q}$ operator lies between the max and min operators, i.e.,

$$(s^-,\alpha^-)\leq 2TLHM^{p,q}\left((s_1,\alpha_1),(s_2,\alpha_2),\cdots,(s_n,\alpha_n)\right)\leq(s^+,\alpha^+).$$

Proof. Since $(s^-,\alpha^-)\leq(s_j,\alpha_j)$, according to monotonicity of $2TLHM^{p,q}$ operator in Theorem 5, we can get:

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}\left(s^{-},\alpha^{-}\right)\right)^{p}\left(\Delta^{-1}\left(s^{-},\alpha^{-}\right)\right)^{q}\right)^{1/p+q}\right)\le$$

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}\left(s_i,\alpha_i\right)\right)^{p}\left(\Delta^{-1}\left(s_j,\alpha_j\right)\right)^{q}\right)^{1/p+q}\right).$$

According to idempotency of $2TLHM^{p,q}$ operator in Theorem 4, we can get:

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}\left(s^{-},\alpha^{-}\right)\right)^{p}\left(\Delta^{-1}\left(s^{-},\alpha^{-}\right)\right)^{q}\right)^{1/p+q}\right)=\left(s^{-},\alpha^{-}\right).$$

So, we can get:

$$\left(s^{-},\alpha^{-}\right)\le\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}\left(s_i,\alpha_i\right)\right)^{p}\left(\Delta^{-1}\left(s_j,\alpha_j\right)\right)^{q}\right)^{1/p+q}\right).$$

Similarly, we can get:

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}\left(s_i,\alpha_i\right)\right)^{p}\left(\Delta^{-1}\left(s_j,\alpha_j\right)\right)^{q}\right)^{1/p+q}\right)\ge\left(s^{+},\alpha^{+}\right).$$

So we have:

$$\left(s^{-},\alpha^{-}\right)\le 2TLHM^{p,q}\left(\left(s_1,\alpha_1\right),\left(s_2,\alpha_2\right),\cdots,\left(s_n,\alpha_n\right)\right)\le\left(s^{+},\alpha^{+}\right).$$

It is easy to prove that $2TLHM^{p,q}$ operator doesn't have the property of commutativity.

Now we can discuss some special cases of the $2TLHM^{p,q}$ operator with respect to the parameters p and q.

(1) When $q\to 0$, the formula (17) reduces to a 2-tuple linguistic generalized linear descending weighted mean (2TLGM) operator, it follows that:

$$\lim_{q\to 0}2TLHB^{p,q}\left(\left(s_1,\alpha_1\right),\left(s_2,\alpha_2\right),\cdots,\left(s_n,\alpha_n\right)\right)=$$

$$\lim_{q\to 0}\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}\left(s_i,\alpha_i\right)\right)^{p}\left(\Delta^{-1}\left(s_j,\alpha_j\right)\right)^{q}\right)^{1/p+q}\right)=$$

$$\Delta\left(\lim_{q\to 0}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}\left(s_i,\alpha_i\right)\right)^{p}\left(\Delta^{-1}\left(s_j,\alpha_j\right)\right)^{q}\right)^{1/p+q}\right)=$$

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\left(n+1-i\right)\left(\Delta^{-1}\left(s_i,\alpha_i\right)\right)^{p}\right)^{1/p}\right). \tag{18}$$

(2) When $p \to 0$, the formula (17) reduces to a 2-tuple linguistic generalized linear ascending weighted mean operator, it follows that:

$$\lim_{p\to 0} 2TLHB^{p,q}\left(\left(s_1,\alpha_1\right),\left(s_2,\alpha_2\right),\cdots,\left(s_n,\alpha_n\right)\right)=\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n} i\left(\Delta^{-1}\left(s_i,\alpha_i\right)\right)^q\right)^{1/q}\right). \quad (19)$$

From (18) and (19), we can know $2TLHM^{p,q}$ operator has the linear weighted function for input data.

(3) When $p=q=\frac{1}{2}$, the formula (17) reduces to a 2-tuple linguistic basic Heronian mean operator, it follows that:

$$2TLHM^{1/2,1/2}\left(\left(s_1,\alpha_1\right),\left(s_2,\alpha_2\right),\cdots,\left(s_n,\alpha_n\right)\right)=\Delta\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{\Delta^{-1}\left(s_i,\alpha_i\right)\Delta^{-1}\left(s_j,\alpha_j\right)}\right). \quad (20)$$

(4) When $p = q = 1$, the formula (17) reduces to a 2-tuple linguistic line Heronian mean operator, it follows that:

$$2TLHM^{1,1}\left(\left(s_1,\alpha_1\right),\left(s_2,\alpha_2\right),\cdots,\left(s_n,\alpha_n\right)\right)=\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}\left(s_i,\alpha_i\right)\right)\left(\Delta^{-1}\left(s_j,\alpha_j\right)\right)\right)^{1/2}\right). \quad (21)$$

The $2TLHM^{p,q}$ operator only considers the input parameters and their interrelationships, and doesn't consider the importance of each input parameter itself. However, in many practical situations, the weight of input data is also an important parameter. So, we can define a 2-tuple linguistic weighted Heronian mean (2*TLWHM*) operator.

Definition 7. Let $a=\left\{\left(s_1,\alpha_1\right),\left(s_2,\alpha_2\right),\cdots,\left(s_n,\alpha_n\right)\right\}$ be a collection of 2-tuple, and $2TLWHM:\Omega^n\to\Omega$, if

$$2TLWHM^{p,q}\left(\left(s_1,\alpha_1\right),\left(s_2,\alpha_2\right),\cdots,\left(s_n,\alpha_n\right)\right)=$$
$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(n\omega_i\Delta^{-1}\left(s_i,\alpha_i\right)\right)^p\left(n\omega_j\Delta^{-1}\left(s_j,\alpha_j\right)\right)^q\right)^{1/(p+q)}\right), \quad (22)$$

where Ω is the set of all 2-tuple, and $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ is the weight vector of $\left(s_j,\alpha_j\right)(j=1,2,\cdots,n)$, $\omega_j\in[0,1],\sum_{j=1}^{n}\omega_j=1$. n is a balance parameter. Then 2*TLWHM* is called the 2-tuple linguistic weighted Heronian mean (2*TLWHM*) operator.

Theorem 7. The 2*TLHM* operator is a special case of the 2*TLWHM* operator.

Proof. When $\omega=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)^T$,

$$2TLWHM^{p,q}\left(\left(s_1,\alpha_1\right),\left(s_2,\alpha_2\right),\cdots,\left(s_n,\alpha_n\right)\right)=$$

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(n\omega_i\Delta^{-1}\left(s_i,\alpha_i\right)\right)^p\left(n\omega_j\Delta^{-1}\left(s_j,\alpha_j\right)\right)^q\right)^{1/p+q}\right)=$$

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(n\frac{1}{n}\Delta^{-1}\left(s_i,\alpha_i\right)\right)^p\left(n\frac{1}{n}\Delta^{-1}\left(s_j,\alpha_j\right)\right)^q\right)^{1/p+q}\right)=$$

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\Delta^{-1}\left(s_i,\alpha_i\right)\right)^p\left(\Delta^{-1}\left(s_j,\alpha_j\right)\right)^q\right)^{1/p+q}\right)=$$

$$2TLHM^{p,q}.$$

Note: It is easy to prove that the $2TLWHM$ operator has the property of monotonicity, but it has not the property of idempotency.

3. An approach to multi-attribute group decision-making method with linguistic information

In this section, we shall propose an approach to MAGDM problems with linguistic information based on the 2-tuple linguistic weighted Heronian mean ($2TLWBM$) operator.

Consider a multiple attribute decision making problem with linguistic information: let $A=\{A_1,A_2,\cdots,A_m\}$ be a discrete set of alternatives, and $C=\{C_1,C_2,\cdots,C_n\}$ be the set of attributes, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ is the weighting vector of the attribute $C_j(j=1,2,\cdots,n)$, where $\omega_j\geq 0, j=1,2,\cdots,n, \sum_{j=1}^{n}\omega_j=1$. Let $D=\{D_1,D_2,\cdots,D_d\}$ be the set of decision makers, and $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_d)$ is the weighting vector of the decision maker, with $\gamma_k\geq 0(k=1,2,\cdots,d), \sum_{k=1}^{d}\gamma_k=1$. Suppose that $X^k=\left[x_{ij}^k\right]_{m\times n}$ is the decision matrix, where x_{ij}^k is an attribute value which is given by decision maker D_k for the alternative $A_i\in A$ with respect to the attribute $C_j\in C$, and it takes the form of linguistic variable, $x_{ij}^k\in S$. Then, the ranking of alternatives is required.

In the following, we apply $2TLWHM$ operator to solve this multiple attribute group decision making problem with linguistic information.

The method involves the following steps:

Step 1. Normalization.

Generally, there are two attribute types in multiple attribute decision making, they are benefit type (the bigger the attribute value is, the better it is) and cost type (the smaller the attribute value is, the better it is), we need a normalization in order to transform the attribute values of the cost type into the attribute values of the benefit type. Suppose $X^k=\left[x_{ij}^k\right]_{m\times n}$ is transformed into the matrix $R^k=\left[r_{ij}^k\right]_{m\times n}$, where

(1) $r_{ij}^k = x_{ij}^k$, for benefit type of C_j;

(2) $r_{ij}^k = Neg(x_{ij}^k)$, for cost type of C_j.

Step 2. Transforming the linguistic decision matrix $R^k = \left[r_{ij}^k\right]_{m\times n}$ into 2-tuple decision matrix $\tilde{R}^k = \left[(r_{ij}^k,0)\right]_{m\times n}$.

Step 3. Utilizing the $2TLWHM$ operator to aggregate all the individual decision matrixes $\tilde{R}^k = \left[(r_{ij}^k,0)\right]_{m\times n}$ $(k=1,2,\cdots,d)$ into the collective decision matrix $\tilde{R} = \left[(r_{ij},\alpha_{ij})\right]_{m\times n}$, where

$$(r_{ij},\alpha_{ij}) = 2TLWHM^{p,q}\left(\left(r_{ij}^0,0\right),\left(r_{ij}^1,0\right),\cdots,\left(r_{ij}^d,0\right)\right) =$$

$$\Delta\left(\left(\frac{2}{d(d+1)}\sum_{k=1}^{d}\sum_{l=k}^{d}\left(d\gamma_k\Delta^{-1}\left(r_{ij}^k,0\right)\right)^p\left(d\gamma_l\Delta^{-1}\left(r_{ij}^l,0\right)\right)^q\right)^{1/(p+q)}\right). \tag{23}$$

Step 4. Utilizing the $2TLWHM$ operator to calculate the comprehensive evaluation value (r_i, α_i) of each alternative, where:

$$\left(r_i,\alpha_i\right) = 2TLWHM^{p,q}\left(\left(r_{i1},\alpha_{i1}\right),\left(r_{i2},\alpha_{i2}\right),\cdots,\left(r_{in},\alpha_{in}\right)\right) =$$

$$\Delta\left(\left(\frac{2}{n(n+1)}\sum_{j=1}^{n}\sum_{k=j}^{n}\left(n\omega_j\Delta^{-1}\left(r_{ij},\alpha_{ij}\right)\right)^p\left(n\omega_k\Delta^{-1}\left(r_{ik},\alpha_{ik}\right)\right)^q\right)^{1/(p+q)}\right). \tag{24}$$

Step 5. Ranking the 2-tuple $\left(r_i,\alpha_i\right)(i=1,2,\cdots,m)$ according to the comparison of 2-tuple in Section 1.1.

Step 6. Ranking all the alternatives $A=\left\{A_1,A_2,\cdots,A_m\right\}$ in accordance with 2-tuple (r_i, α_i) in descending order, and then select the most desirable alternative with the largest overall performance value.

Step 7. End.

4. Numerical example

Suppose an investment company wants to invest a sum of money in the best option (adapted from Xu and Wang (2011)). There is a panel with four possible alternatives in which to invest the money: (1) A_1 is a car industry; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investment company must take a decision according to the following three attributes: (1) C_1 is the risk analysis; (2) C_2 is the growth analysis; (3) C_3 is the social-political impact analysis. Suppose that the weight vector of three attributes is $\omega = \left(0.3,0.4,0.3\right)^T$. The four possible alternatives $A_i (i=1,2,3,4)$ are evaluated by three decision makers under the above four attributes and construct, respectively, the decision matrix $X^k = \left[x_{ij}^k\right]_{4\times 3}$ $(k=1,2,3)$ which is listed in Tables 1–3, where $x_{ij}^k \in S$, $S=(s_0,s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8)$ = {extremely poor, very poor, poor, slightly poor, fair, slightly good, good, very good, extremely good}. Suppose that the weight vector of three decision makers is $\gamma = \left(0.35,0.25,0.4\right)^T$. Try to determine the best investment alternative.

Table 1. Linguistic decision matrix X^1

	C_1	C_2	C_3
$A1$	s_6	s_6	s_7
A_2	s_6	s_4	s_5
A_3	s_3	s_4	s_6
A_4	s_6	s_4	s_3

Table 2. Linguistic decision matrix X^2

	C_1	C_2	C_3
A_1	s_4	s_5	s_4
A_2	s_7	s_6	s_5
A_3	s_4	s_5	s_6
A_4	s_5	s_4	s_5

Table 3. Linguistic decision matrix X^3

	C_1	C_2	C_3
A_1	s_3	s_5	s_4
A_2	s_7	s_6	s_5
A_3	s_5	s_4	s_7
A_4	s_7	s_6	s_5

4.1. Decision-making steps

Step 1. Normalization.

Because all attribute values are benefit type, we can get $R^k = \left[r_{ij}^k\right]_{m\times n} = \left[x_{ij}^k\right]_{m\times n}$.

Step 2. Transforming linguistic decision matrices $R^k = \left[r_{ij}^k\right]_{m\times n} (k = 1,2,3)$ into 2-tuple decision matrices $\tilde{R}^k = \left[(r_{ij}^k, 0)\right]_{m\times n}$ which are shown in Tables 4–6.

Table 4. 2-tuple linguistic decision matrix $\tilde{R}^1$

	C_1	C_2	C_3
A_1	$(s_6, 0)$	$(s_6, 0)$	$(s_7, 0)$
A_2	$(s_6, 0)$	$(s_4, 0)$	$(s_5, 0)$
A_3	$(s_3, 0)$	$(s_4, 0)$	$(s_6, 0)$
A_4	$(s_6, 0)$	$(s_4, 0)$	$(s_3, 0)$

Table 5. 2-tuple linguistic decision matrix $\tilde{R}^2$

	C_1	C_2	C_3
A_1	$(s_4, 0)$	$(s_5, 0)$	$(s_4, 0)$
A_2	$(s_7, 0)$	$(s_6, 0)$	$(s_5, 0)$
A_3	$(s_4, 0)$	$(s_5, 0)$	$(s_6, 0)$
A_4	$(s_5, 0)$	$(s_4, 0)$	$(s_5, 0)$

Table 6. 2-tuple linguistic decision matrix $\tilde{R}^3$

	C_1	C_2	C_3
A_1	$(s_3, 0)$	$(s_5, 0)$	$(s_4, 0)$
A_2	$(s_7, 0)$	$(s_6, 0)$	$(s_5, 0)$
A_3	$(s_5, 0)$	$(s_4, 0)$	$(s_7, 0)$
A_4	$(s_7, 0)$	$(s_6, 0)$	$(s_5, 0)$

Step 3. Utilizing the $2TLWHM$ operator expressed in (23) to aggregate all the individual decision matrixes $\tilde{R}^k = \left[(r_{ij}^k, 0)\right]_{4\times 3} (k = 1,2,3)$ into the collective decision matrix $\tilde{R} = \left[(r_{ij}, \alpha_{ij})\right]_{4\times 3}$, we can get (suppose $p = q = 1$):

$$\tilde{R} = \begin{bmatrix} (s_4, 0.36) & (s_5, 0.38) & (s_5, 0.13) \\ (s_7, -0.32) & (s_5, 0.34) & (s_5, 0.02) \\ (s_4, 0.11) & (s_4, 0.26) & (s_6, 0.45) \\ (s_6, 0.22) & (s_5, -0.12) & (s_4, 0.34) \end{bmatrix}.$$

Step 4. Utilizing the $2TLWHM$ operator expressed in (24) to calculate the comprehensive evaluation value $\tilde{r}_i = \ r_i, \alpha_i$ of each alternative, we can get (suppose $p = q = 1$):

$$\tilde{r}_1 = \left(s_5, 0.03\right), \tilde{r}_2 = \left(s_6, -0.34\right), \tilde{r}_3 = \left(s_5, -0.11\right), \tilde{r}_4 = \left(s_5, 0.14\right).$$

Step 5. Ranking the 2-tuples $\tilde{r}_i = \left(r_i, \alpha_i\right)(i = 1,2,3,4)$, we can get:

$$\tilde{r}_2 > \tilde{r}_4 > \tilde{r}_1 > \tilde{r}_3.$$

Step 6. Ranking all the alternatives $A = \left\{A_1, A_2, A_3, A_4\right\}$ in accordance with $\tilde{r}_i$, we can get:

$$A_2 \succ A_4 \succ A_1 \succ A_3.$$

Thus the best alternative is A_2.

This result is the same as that in Xu and Wang (2011), and it verifies the validity of this method.

4.2. Discussion

In order to illustrate the influence of the parameters p, q on decision making of this example, we use the different values p, q in step (3) and (4) to rank the alternatives. The ranking results are shown in Table 7.

Table 7. Ordering of the alternatives by utilizing the different p, q in $2TLWHM$ operator

p, q	Comprehensive values $\tilde{r}_i (i=1,2,3,4)$	Ranking
$p=q=\frac{1}{2}$	$\tilde{r}_1=(s_5,-0.18)$, $\tilde{r}_2=(s_6,-0.46)$ $\tilde{r}_3=(s_5,-0.25)$, $\tilde{r}_4=(s_5,-0.07)$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$p=1, q=0$	$\tilde{r}_1=(s_5,0.12)$, $\tilde{r}_2=(s_5,0.49)$ $\tilde{r}_3=(s_4,0.18)$, $\tilde{r}_4=(s_5,-0.10)$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p=0, q=1$	$\tilde{r}_1=(s_5,-0.22)$, $\tilde{r}_2=(s_6,-0.32)$ $\tilde{r}_3=(s_5,0.46)$, $\tilde{r}_4=(s_5,0.24)$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$p=2, q=1$	$\tilde{r}_1=(s_5,0.32)$, $\tilde{r}_2=(s_6,-0.26)$ $\tilde{r}_3=(s_5,-0.17)$, $\tilde{r}_4=(s_5,0.31)$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p=1, q=2$	$\tilde{r}_1=(s_5,0.19)$, $\tilde{r}_2=(s_6,-0.14)$ $\tilde{r}_3=(s_5,0.24)$, $\tilde{r}_4=(s_5,0.47)$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$p=2, q=2$	$\tilde{r}_1=(s_5,0.40)$, $\tilde{r}_2=(s_6,-0.09)$ $\tilde{r}_3=(s_5,0.14)$, $\tilde{r}_4=(s_6,-0.44)$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$p=1, q=5$	$\tilde{r}_1=(s_6,-0.23)$, $\tilde{r}_2=(s_6,0.48)$ $\tilde{r}_3=(s_6,-0.13)$, $\tilde{r}_4=(s_6,0.32)$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$p=5, q=1$	$\tilde{r}_1=(s_6,-0.06)$, $\tilde{r}_2=(s_6,0.12)$ $\tilde{r}_3=(s_5,0.10)$, $\tilde{r}_4=(s_6,-0.04)$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$p=5, q=5$	$\tilde{r}_1=(s_6,0.11)$, $\tilde{r}_2=(s_7,-0.39)$ $\tilde{r}_3=(s_6,-0.26)$, $\tilde{r}_4=(s_7,-0.46)$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$p=30, q=30$	$\tilde{r}_1=(s_7,0.15)$, $\tilde{r}_2=(s_8,0.141574)$ $\tilde{r}_3=(s_7,0.12)$, $\tilde{r}_4=(s_8,0.141573)$	$A_2 \succ A_4 \succ A_1 \succ A_3$

As we can see from Table 7, the ordering of the alternatives may be different for the different values p, q in $2TLWHM$ operator, and the higher the values p and q are, the higher the comprehensive values $\tilde{r}_i (i=1,2,3,4)$ are. However, the best selection is A_2 in this example. In general, we can get $p = q = 1$ or $p = q = 1/2$. In some special cases which need a linear weighting, we can select $p = 0$ or $q = 0$, because they have the function of the linear weighting.

Conclusions

The traditional Heronian mean operators are generally suitable for aggregating the information with crisp numbers, and yet they will fail in dealing with linguistic variables. In this paper, we have developed some new linguistic Heronian mean aggregation operators, such as 2-tuple linguistic Heronian mean (2TLHM) operator, and 2-tuple linguistic weighted Heronian mean (2TLWHM) operator which are based on Heronian mean operator. Furthermore, we have studied some desired properties of these operators, such as idempotency, monotonicity, boundedness, and discussed some special cases with respect to the parameters in these operators. Moreover, with respect to MAGDM problems in which both the attribute weights and the expert weights take the form of real numbers, attribute values take the form of linguistic variables, an approach based on the developed operators is proposed. The prominent characteristic of the developed approach is that they can take all the decision arguments and their relationships into account. Finally, an illustrative example has been given to show the steps of the developed method and to discuss the influences of different parameters on the decision-making results.

In the future, on the one hand, we will further research the applications of the proposed method, and on the other hand, we will investigate the operators to uncertain linguistic variables, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, etc. or further generalize these operators by using the well-known Choquet integral and Dempster-Shafer belief structure, or extend the potential applications of the developed linguistic aggregation operators to other domains, such as pattern recognition, supply chain management, etc.

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