



INTUITIONISTIC FUZZY INTERACTION MACLAURIN SYMMETRIC MEANS AND THEIR APPLICATION TO MULTIPLE-ATTRIBUTE DECISION-MAKING

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Abstract. The Maclaurin symmetric mean (MSM) can capture the interrelationship among the multi-input arguments and it also can generalize most of the existing operators. Now MSM has been extended to intuitionistic fuzzy sets (IFSs) which can easily express the vague information. However, the operational rules of IFSs used in the extended MSM operator didn't consider the interaction between the membership function and non-membership function, so there are some weaknesses. In this paper, in order to combine the advantages of the MSM and interaction operational rules of IFSs, we propose the intuitionistic fuzzy interaction Maclaurin symmetric mean (IFIMSM) operator, the intuitionistic fuzzy weighted interaction Maclaurin symmetric mean (IFWIMSM) operator, respectively. Furthermore, we research some desirable properties and some special cases of them. Further, we develop a new method to deal with some multi-attribute group decision-making (MAGDM) problems under intuitionistic fuzzy environment based on these operators. Finally, an illustrative example is given to testify the availability of the developed method by comparing with the other existing methods.

Keywords: intuitionistic fuzzy set, Maclaurin symmetric mean operator, multi-attribute group decision-making.

JEL Classification: C44, C60.

Introduction

Now MAGDM has been applied in all kind of fields more and more widely (Celik, Gumus, & Alegoz, 2014; Gürbüz & Albayrak, 2014; Mulliner, Malys, & Maliene, 2015; Rabbani, Zamani, Yazdani-Chamzini, & Zavadskas, 2014; Zhang, & Guo, 2016; Wu, Cao, & Li, 2016; Tian, Wang, & Wang, 2017), the goal of these decision-makings is to select the best one from the finite alternatives according to some attributes. So it is very important for some individuals and enterprises to make a reasonable decision. Because the complexity of decision-making

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problems and decision-making environments, it is usually difficult to describe attribute values of alternatives by real values. Zadeh (1965) proposed the theory of fuzzy sets (FSs) firstly, which provided a convenient tool to express fuzzy information. Whereas, because FS only has the membership function, so it is difficult to describe some complex fuzzy information sometimes, then Atanassov (1986, 1989) proposed the intuitionistic FSs (IFSs) to express more complex fuzzy information, which are composed of a membership function and a non-membership function. Obviously, IFS can deal with the vague information more accurately and effectively than the traditional FS. In recent years, the theory of IFSs has been widely studied and a great deal of achievements have been made, such as operations on IFSs (De Kumar, Biswas, & Roy, 2000a), distance measures between IFSs (Chen, 2007) and applications in MAGDM problems (Li, 2005; Lin, Yuan, & Xia, 2007; Xu, Huang, Da, & Liu, 2010; Xu, 2011; Zhang, & Xu, 2015; Zhou, Wang, & Zhang, 2016; Uygun & Dede, 2016).

In recent years, information aggregation operators have attracted wide attentions of researchers and have become an important research topic for MAGDM problems (Şahin & Liu, 2017; Straub & Reza, 2015). Because they have more advantages than traditional approaches such as TOPSIS (Wang, Liu, Li, & Niu, 2016b), VIKOR (Tavana, Mavi, Santos-Arteaga, & Doust, 2016), TODIM (Liu & Teng, 2016; J. Wang, J. Q. Wang, & Zhang, 2016a), PROMETHE (Montajabiha, 2016) and so on. Aggregation operators can provide the comprehensive values of the alternatives and then give the rankings on the basis of them, while traditional approaches can only give the ranking results. In general, we consider information aggregation operators from two aspects: the functions and the operations.

(1) For the functions, the traditional aggregation operators (Xu, 2007; Xu & Yager, 2006) only can aggregate a set of real values into one. Now some extended aggregation operators have been developed for some special functions. Meng, Zhang, and Zhan (2015) proposed the Choquet aggregation operator for intuitionistic fuzzy numbers (IFNs), which considers the interaction among aggregating parameters; Wang, Zeng, and Zhang (2013) proposed dependent aggregation operators for IFNs which can relieve the influences of unreasonable data by dependent weights; Xu and Yager (2011) proposed Bonferroni mean (BM) operators for IFNs and Yu and Wu (2012) proposed Heronian mean (HM) operators for interval-valued IFNs (IVIFNs) which all can consider interrelationships between aggregating parameters. Then researches on BM and HM operators have made some achievements (P. D. Liu, Chen, & J. L. Liu, 2017; Liu, & Li, 2017; P. D. Liu, J. L. Liu, & Chen, 2018; Liu, 2017; Liu, & Chen, 2017; P. D. Liu, J. L. Liu, & Merigó, 2018). However, the BM and HM operators just can consider the interrelationship between two arguments, further, Qin and Liu (2014) proposed Maclaurin symmetric mean (MSM) operators for IFNs which consider the interrelationship among any multi-input arguments by a variable parameter, so it can be more adequate to solve the MAGDM problems by considering the interrelationships.

(2) For the operations, the traditional operations of IFSs cannot consider the interactions between membership function and non-membership function, so in some special cases, they get unreasonable aggregating results, especially when there exist zero in non-membership of IFSs. For example, let $\tilde{b}_1 = (u_1, v_1)$, $\tilde{b}_2 = (u_2, v_2)$ be two IFNs, and $v_1 \neq 0, v_2 = 0$, then, by the addition operation by Atanassov (1994), then we can get $v_1 \otimes v_2 = 0$. In other word, no matter what value v_1 is, the result of $v_1 \otimes v_2$ is still zero because of $v_2 = 0$, which is an undesirable property. To solve this shortcoming, He, Chen, Zhou, Liu, & Tao (2014b) proposed

the interaction operational rules of IFNs, which can take account of the interactions between membership function and non-membership function sufficiently and overcome the problem when the non-membership degree of IFNs is zero.

Because the MSM can consider the interrelationship of any multiple attributes, it is more general to capture the interrelationship than BM and HM operators because the BM or HM can only consider two attributes. In addition, IFSs can easily express the complex fuzzy information. However, because the traditional operational rules of IFSs proposed by Atanassov (1989) do not consider the interactions between membership function and non-membership function, in some situations, they may get the unreasonable results, especially when the non-membership degree is zero. So the aim of this paper is to combine the MSM with the IFSs with the interaction operational rules, and to develop some interaction MSM operators for IFNs to overcome the weaknesses of the existing operators based on the new operational rules on IFNs by He, Chen, Zhou, Han, and Zhao (2014a), He et al. (2014b). The advantages of new proposed operators are that they not only consider the superiorities of MSM, but also consider the interaction relationship between the membership function and non-membership function of IFSs.

The rest of this paper is organized as follows. In Section 1, we briefly review the basic concepts of IFSs, the MSM operator and the new improved interactional operations rules. In Section 2, we propose some interaction MSM operators for IFNs on the basis of the interaction rules. In Section 3, we propose a MAGDM method with IFNs based on the proposed IFWIMSM operators. In Section 4, we use some examples to illustrate the effectiveness of the proposed new method. The conclusions are discussed in last Section.

1. Preliminaries

In this section, some basic concepts were introduced, including the concept and basic operations of IFS, the interactional operations and the MSM operator.

1.1. IFS

Definition 1 (Atanassov, 1986). Let a set Z be fixed, an IFS \tilde{b} in Z is given by

$$\tilde{b} = \left\{ \langle z, u_{\tilde{b}}(z), v_{\tilde{b}}(z) \rangle \mid z \in Z \right\}, \tag{1}$$

where $u_{\tilde{b}}(z)$ is the membership function, and $v_{\tilde{b}}(z)$ is the non-membership function. For each point z in Z , we have $\mu_{\tilde{b}}(z) \in [0,1], v_{\tilde{b}}(z) \in [0,1]$ and $0 \leq \mu_{\tilde{b}}(z) + v_{\tilde{b}}(z) \leq 1, \forall z \in Z$.

In addition, we call $\pi_{\tilde{b}}(z) = 1 - \mu_{\tilde{b}}(z) - v_{\tilde{b}}(z)$ a hesitancy degree which can meet $0 \leq \pi_{\tilde{b}}(z) \leq 1, \forall z \in Z$ (Atanassov, 1986, 1989).

To the given element z , each pair of $(\mu_{\tilde{b}}(z), v_{\tilde{b}}(z))$ in \tilde{b} is called an IFN (Xu, 2007). For convenience, we use $\tilde{b} = (u_{\tilde{b}}, v_{\tilde{b}})$ to represent an IFN, which meets $u_{\tilde{b}} \in [0,1], v_{\tilde{b}} \in [0,1]$ and $0 \leq u_{\tilde{b}} + v_{\tilde{b}} \leq 1$.

Definition 2 (Chen & Tan, 1994; Hong & Choi, 2000). Let $\tilde{b} = (u_{\tilde{b}}, v_{\tilde{b}})$ be an IFN, then the score function S of \tilde{b} can be defined as follows:

$$S(\tilde{b}) = u_{\tilde{b}} - v_{\tilde{b}}, \tag{2}$$

where $S(\tilde{b}) \in [-1,1]$.

and the accuracy function H of \tilde{b} is defined as follows:

$$H(\tilde{b}) = u_{\tilde{b}} + v_{\tilde{b}}, \tag{3}$$

where $H(\tilde{b}) \in [0,1]$, and the larger the accuracy degree $H(\tilde{b})$ is, the greater \tilde{b} is.

On the basis of the score function and accuracy function, Xu and Yager (2006) developed a comparison method of IFNs, which can be defined as follows:

- (1) If $S(\tilde{b}_1) > S(\tilde{b}_2)$, then, $\tilde{b}_1 \succ \tilde{b}_2$;
- (2) If $S(\tilde{b}_1) = S(\tilde{b}_2)$, then,
If $H(\tilde{b}_1) > H(\tilde{b}_2)$, then $\tilde{b}_1 \succ \tilde{b}_2$;
- If $H(\tilde{b}_1) = H(\tilde{b}_2)$, then $\tilde{b}_1 = \tilde{b}_2$.

Definition 3 (Atanassov, 1994; De Kumar, Biswas, & Roy, 2000a; De Kumar, Biswas, & Roy, 2000b). Let $\tilde{b}_1 = (u_1, v_1)$ and $\tilde{b}_2 = (u_2, v_2)$ be any two IFNs and $\lambda > 0$, then

$$(1) \tilde{b}_1 \oplus \tilde{b}_2 = (u_1 + u_2 - u_1 u_2, v_1 v_2), \tag{4}$$

$$(2) \lambda \tilde{b}_1 = (1 - (1 - u_1)^\lambda, v_1^\lambda), \tag{5}$$

$$(3) \tilde{b}_1 \otimes \tilde{b}_2 = (u_1 u_2, v_1 + v_2 - v_1 v_2), \tag{6}$$

$$(4) \tilde{b}_1^\lambda = (u_1^\lambda, 1 - (1 - v_1)^\lambda). \tag{7}$$

However, the traditional operational rules in Definition 3 have some weaknesses.

Example 1. Suppose $A_1 = (0.5, 0.3)$, $A_2 = (0.6, 0.2)$, $A_3 = (0.5, 0)$, and the weight vector of them is $\omega = (0.35, 0.40, 0.25)$. Then by Definition 3, we obtain $A = \oplus \omega_i A_i = (0.5427, 0)$. Obviously, $v_A = v_{A_3} = 0$, in other words, v_{A_j} ($j=1, 2$) have no effects on the overall result, which is an undesirable property. In addition, the traditional operational rules also ignore the interaction between the membership and non-membership.

In order to solve these problems, He, Chen, Zhou, Liu, and Tao (2014b) proposed the improved rules that consider the interaction between the membership function and the non-membership function of different IFNs, which are defined as follows.

1.2. The improved operations

Definition 4 (He et al., 2014a, 2014b). Let $\tilde{b}_1 = (u_1, v_1)$ and $\tilde{b}_2 = (u_2, v_2)$ be any two IFNs and $\lambda > 0$, and then the interactional operational rules of IFNs are defined as follows:

$$1) \tilde{b}_1 \oplus \tilde{b}_2 = (1 - (1 - u_1)(1 - u_2), (1 - u_1)(1 - u_2) - (1 - (u_1 + v_1))(1 - (u_2 + v_2))); \tag{8}$$

$$2) \lambda \tilde{b}_1 = (1 - (1 - u_1)^\lambda, (1 - u_1)^\lambda - (1 - (u_1 + v_1))^\lambda); \tag{9}$$

$$3) \tilde{b}_1 \otimes \tilde{b}_2 = ((1 - v_1)(1 - v_2) - (1 - (u_1 + v_1))(1 - (u_2 + v_2)), 1 - (1 - v_1)(1 - v_2)); \tag{10}$$

$$4) \tilde{b}_1^\lambda = ((1 - v_1)^\lambda - (1 - (u_1 + v_1))^\lambda, 1 - (1 - v_1)^\lambda). \tag{11}$$

Theorem 1 (He et al., 2014b). Suppose $\tilde{b}_1 = (u_1, v_1)$ and $\tilde{b}_2 = (u_2, v_2)$ are any two IFNs, and $\lambda, \lambda_1, \lambda_2 > 0$, then the interactional operational rules of IFNs meet the properties as follows.

$$(1) \tilde{b}_1 \otimes \tilde{b}_2 = \tilde{b}_2 \otimes \tilde{b}_1; \tag{12}$$

$$(2) (\tilde{b}_1 \otimes \tilde{b}_2)^\lambda = \tilde{b}_1^\lambda \otimes \tilde{b}_2^\lambda; \tag{13}$$

$$(3) \tilde{b}_1^{\lambda_1} \otimes \tilde{b}_1^{\lambda_2} = \tilde{b}_1^{(\lambda_1+\lambda_2)}; \tag{14}$$

Example 2. If we solve the Example 1 with the improved operational rules in Definition 4, then we obtain $A' = \omega_1 A_1 \oplus \omega_2 A_2 \oplus \omega_3 A_3 = (0.4562, 0.2058)$. Obviously, this result is more reasonable. Therefore, the improved operational rules are more practical in some cases, because of considering the interaction of different IFNs.

1.3. MSM Operators

The MSM, originally introduced by Maclaurin (1729), is a useful technique to capture the interrelationship among the multi-input arguments, which is given as follows.

Definition 5 (Maclaurin, 1729). Let $a_i (i = 1, 2, \dots, n)$ be a collection of nonnegative real numbers, and $k = 1, 2, \dots, n$ is a parameter, the MSM is defined as

$$MSM^{(k)}(a_1, a_2, \dots, a_n) = \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k a_{i_j}}{C_n^k} \right)^{1/k}, \tag{15}$$

where (i_1, i_2, \dots, i_k) traverses all the k -tuple combination of $(1, 2, \dots, n)$ and $C_n^k = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

Obviously, the MSM has the following properties:

- (1) $MSM^{(k)}(0, 0, \dots, 0) = 0, MSM^{(k)}(a, a, \dots, a) = a;$
- (2) $MSM^{(k)}(a_1, a_2, \dots, a_n) \leq MSM^{(k)}(b_1, b_2, \dots, b_n),$ if $a_i \leq b_i$ for all $i;$
- (3) $\min_i \{a_i\} \leq MSM^{(k)}(a_1, a_2, \dots, a_n) \leq \max_i \{a_i\}.$

2. Intuitionistic fuzzy interaction MSM operators

In this section, on the basis of the improved operational rules of IFNs, we propose the interaction MSM operator for IFNs (IFIMSM) and weighted interaction MSM operator for IFNs (IFWIMSM), and then we will investigate some special cases and properties.

2.1. IFIMSM operator

Definition 6. Let $\tilde{b}_i (i = 1, 2, \dots, n)$ be a collection of IFNs, and $k = 1, 2, \dots, n$ is a parameter, then the IFIMSM operator is a mapping $IFIMSM : \Phi^n \rightarrow \Phi$ defined as follows:

$$IFIMSM^{(k)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \left(\frac{\bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \bigotimes_{j=1}^k \tilde{b}_{i_j}}{C_n^k} \right)^{1/k}, \tag{16}$$

where Φ is the set of all IFNs, (i_1, i_2, \dots, i_k) traverses all the k -tuple combination of $(1, 2, \dots, n)$ and $C_n^k = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

Based on the improved operational rules of the IFNs, we can derive the result shown as theorem 2.

Theorem 2. Let $\tilde{b}_i = (u_i, v_i)$ ($i = 1, 2, \dots, n$) be a collection of IFNs and $k = 1, 2, \dots, n$, then, the result aggregated from (16) is still an IFN, and

$$\begin{aligned}
 IFIMSM^{(k)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = & \left(\left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - v_{i_j}) + \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} - \right. \\
 & \left. \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}}, \\
 1 - & \left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - v_{i_j}) + \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{k}}.
 \end{aligned} \tag{17}$$

Proof.

Firstly, we can calculate $\bigotimes_{j=1}^k \tilde{b}_{i_j}$, and obtain

$$\begin{aligned}
 \bigotimes_{j=1}^k \tilde{b}_{i_j} &= \left(\prod_{j=1}^k (1 - v_{i_j}) - \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}), 1 - \prod_{j=1}^k (1 - v_{i_j}) \right), \\
 \text{and } \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \left(\bigotimes_{j=1}^k \tilde{b}_{i_j} \right) &= \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - v_{i_j}) + \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right), \right. \\
 & \left. \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - v_{i_j}) + \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right) - \prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right),
 \end{aligned}$$

then we get

$$\begin{aligned}
 \frac{1}{C_n^k} \left(\bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \left(\bigotimes_{j=1}^k \tilde{b}_{i_j} \right) \right) &= \left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - v_{i_j}) + \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{1}{C_n^k}}, \right. \\
 & \left. \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - v_{i_j}) + \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{1}{C_n^k}} - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}}.
 \end{aligned}$$

Therefore, we have

$$\left(\frac{\bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \bigotimes_{j=1}^k \tilde{b}_{i_j}}{C_n^k} \right)^{1/k} = \left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - v_{i_j}) + \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}},$$

$$1 - \left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - v_{i_j}) + \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}}.$$

So, the theorem 2 is proved.

Example 3: Let $\tilde{b}_1 = (0.5, 0.3)$, $\tilde{b}_2 = (0.6, 0.2)$ and $\tilde{b}_3 = (0.5, 0)$ be three IFNs, then we can use the IFISM operator to aggregate the three IFNs as follows.

Calculate the comprehensive value $\tilde{b} = (u, v)$ by Eq. (17) (without loss of generality, we suppose $k = 2$), and we get

$$\tilde{b}_1 \otimes \tilde{b}_2 = \langle 0.5 + 0.6 - 0.5 \times 0.6 - 0.5 \times 0.2 - 0.3 \times 0.6, 0.3 + 0.2 - 0.3 \times 0.2 \rangle = (0.52, 0.44),$$

$$\tilde{b}_1 \otimes \tilde{b}_3 = \langle 0.5 + 0.5 - 0.5 \times 0.5 - 0.5 \times 0 - 0.3 \times 0.5, 0.3 + 0 - 0.3 \times 0 \rangle = (0.6, 0.3),$$

$$\tilde{b}_2 \otimes \tilde{b}_3 = \langle 0.6 + 0.5 - 0.6 \times 0.5 - 0.6 \times 0 - 0.2 \times 0.5, 0.2 + 0 - 0.2 \times 0 \rangle = (0.7, 0.2),$$

and

$$IFISM^{(2)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = \left(\frac{\bigoplus_{1 \leq i_1 < i_2 \leq 3} \bigotimes_{j=1}^2 \tilde{b}_{i_j}}{C_3^2} \right)^{1/2} = \left(\frac{(\tilde{b}_1 \otimes \tilde{b}_2) \oplus (\tilde{b}_2 \otimes \tilde{b}_3) \oplus (\tilde{b}_1 \otimes \tilde{b}_3)}{3} \right)^{1/2} = (0.558, 0.171).$$

It is easy to prove that the IFISM operator has the following properties.

Theorem 3 (Idempotency). Let $\tilde{b}_j = (u_j, v_j) (j=1, 2, \dots, n)$ be a collection of IFNs, if $\tilde{b}_j = \tilde{b} = (u, v), j=1, 2, \dots, n$, then

$$IFISM = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \tilde{b} = (u, v). \tag{18}$$

Proof.

Since $\tilde{b}_j = (u, v) (j=1, 2, \dots, n)$, then based on formula (17), we have

$$\begin{aligned}
 IFIMSM^{(k)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = & \left(\left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1-v) + \prod_{j=1}^k (1-u-v) \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1-u-v) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} - \right. \\
 & \left. \left(\left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1-u-v) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right) \\
 & \left(1 - \left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1-v) + \prod_{j=1}^k (1-u-v) \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1-u-v) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right) = \\
 & \left(\left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - (1-v)^k + (1-u-v)^k \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} (1-u-v)^k \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} - \right. \\
 & \left. \left(\left(\prod_{1 \leq i_1 < \dots < i_k \leq n} (1-u-v)^k \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right) \\
 & \left(1 - \left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - (1-v)^k + (1-u-v)^k \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} (1-u-v)^k \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right) = (u, v) = \tilde{b}.
 \end{aligned}$$

Theorem 4 (Commutativity). Suppose $\tilde{b}_j = (u_j, v_j) (j=1, 2, \dots, n)$ is a set of IFNs, and $\tilde{b}'_j = (u'_j, v'_j)$ is any permutation of \tilde{b}_j , then

$$IFIMSM^{(k)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = IFIMSM^{(k)}(\tilde{b}'_1, \tilde{b}'_2, \dots, \tilde{b}'_n). \tag{19}$$

Proof.

On the basis of Eq. (16), we have

$$IFIMSM^{(k)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \left(\frac{\bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \bigotimes_{j=1}^k \tilde{b}_{i_j}}{C_n^k} \right)^{1/k},$$

$$IFIMSM^{(k)}(\tilde{b}'_1, \tilde{b}'_2, \dots, \tilde{b}'_n) = \left(\frac{\bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \bigotimes_{j=1}^k \tilde{b}'_{i_j}}{C_n^k} \right)^{\frac{1}{k}},$$

Since $\tilde{b}'_j = (u'_j, v'_j)$ be any permutation of \tilde{b}_j , then

$$\left(\frac{\bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \bigotimes_{j=1}^k \tilde{b}'_{i_j}}{C_n^k} \right)^{1/k} = \left(\frac{\bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \bigotimes_{j=1}^k \tilde{b}_{i_j}}{C_n^k} \right)^{1/k}.$$

Thus, $IFIMSM^{(k)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = IFIMSM^{(k)}(\tilde{b}'_1, \tilde{b}'_2, \dots, \tilde{b}'_n)$.

Now we can discuss some special cases of the IFIMSM operator based on different values of the parameter k .

(1) When $k = 1$, based on the IFIMSM operator (17), we have

$$IFIMSM^{(1)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \left(\left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^1 (1 - v_{i_j}) + \prod_{j=1}^1 (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{1}{C_n^1}} + \left(\prod_{1 \leq i_1 \leq n} \prod_{j=1}^1 (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}} - \left(\prod_{1 \leq i_1 \leq n} \prod_{j=1}^1 (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}},$$

$$= 1 - \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^1 (1 - v_{i_j}) + \prod_{j=1}^1 (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{1}{C_n^1}} + \left(\prod_{1 \leq i_1 \leq n} \prod_{j=1}^1 (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}} =$$

$$\left(1 - \left(\prod_{1 \leq i_1 \leq n} (1 - u_{i_j}) \right)^{\frac{1}{n}}, \left(\prod_{1 \leq i_1 \leq n} (1 - u_{i_j}) \right)^{\frac{1}{n}} - \left(\prod_{1 \leq i_1 \leq n} (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{n}} \right) \text{ (let } i_j = j) =$$

$$\left(1 - \left(\prod_{j=1}^n (1 - u_j) \right)^{\frac{1}{n}}, \left(\prod_{j=1}^n (1 - u_j) \right)^{\frac{1}{n}} - \left(\prod_{j=1}^n (1 - u_j - v_j) \right)^{\frac{1}{n}} \right). \tag{20}$$

In this case, the IFIMSM reduces to the intuitionistic fuzzy interaction averaging (IFIA) operator.

(3) If $k = 2$, then the IFIMSM operator (17) will reduce to the intuitionistic fuzzy interaction BM (IFIBM) operator ($p = q = 1$). It is shown as follows:

$$\begin{aligned}
 &IFIMSM^{(2)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \\
 &\left(\left(1 - \left(\prod_{1 \leq i_1 < i_2 \leq n} \left(1 - \prod_{j=1}^2 (1 - v_{i_j}) + \prod_{j=1}^2 (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{2}{n(n-1)}} + \left(\prod_{1 \leq i_1 < i_2 \leq n} \prod_{j=1}^2 (1 - u_{i_j} - v_{i_j}) \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}} - \right. \\
 &\left. \left(\prod_{1 \leq i_1 < i_2 \leq n} \prod_{j=1}^2 (1 - u_{i_j} - v_{i_j}) \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}}, \\
 &1 - \left(1 - \left(\prod_{1 \leq i_1 < i_2 \leq n} \left(1 - \prod_{j=1}^2 (1 - v_{i_j}) + \prod_{j=1}^2 (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{2}{n(n-1)}} + \left(\prod_{1 \leq i_1 < i_2 \leq n} \prod_{j=1}^2 (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} = \\
 &IFIBM^{1,1}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n). \tag{21}
 \end{aligned}$$

(4) If $k = n$, according to the IFIMSM operator (17), we have

$$\begin{aligned}
 &IFIMSM^{(n)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \\
 &\left(\left(1 - \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \left(1 - \prod_{j=1}^n (1 - v_{i_j}) + \prod_{j=1}^n (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{1}{C_n^n}} + \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^n}} \right)^{\frac{1}{n}} - \right. \\
 &\left. \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^n}} \right)^{\frac{1}{n}}, \\
 &1 - \left(1 - \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \left(1 - \prod_{j=1}^n (1 - v_{i_j}) + \prod_{j=1}^n (1 - u_{i_j} - v_{i_j}) \right) \right)^{\frac{1}{C_n^n}} + \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{C_n^n}} \right)^{\frac{1}{n}} = \\
 &\left(\left(1 - \prod_{1 \leq i_1 < \dots < i_n \leq n} \left(1 - \prod_{j=1}^n (1 - v_{i_j}) + \prod_{j=1}^n (1 - u_{i_j} - v_{i_j}) \right) \right) + \prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{n}} -
 \end{aligned}$$

$$\left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{n}},$$

$$1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_n \leq n} \left(1 - \prod_{j=1}^n (1 - v_{i_j}) + \prod_{j=1}^n (1 - u_{i_j} - v_{i_j}) \right) + \prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n (1 - u_{i_j} - v_{i_j}) \right)^{\frac{1}{n}}. \quad (22)$$

2.2. IFWIMSM operator

In section 2.1, it is clear that the IFIMSM operator does not consider the importance of the attribute weights. Nevertheless, in many practical situations, especially in MAGDM problems, the weights of input arguments play an important role for decision-making results. In order to overcome the limitation of IFIMSM operator, we develop the IFWIMSM operator as follows.

Definition 7. Suppose $\tilde{b}_i (i = 1, 2, \dots, n)$ is a collection of IFNs, and $k = 1, 2, \dots, n$ is a parameter, $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of $\tilde{b}_i (i = 1, 2, \dots, n)$, then the IFWIMSM operator is a mapping $IFWIMSM : \Phi^n \rightarrow \Phi$ defined as follows:

$$IFWIMSM^{(k)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \left(\frac{\bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \bigotimes_{j=1}^k (w_{i_j} \tilde{b}_{i_j})}{C_n^k} \right)^{1/k}, \quad (23)$$

where Φ is the set of all IFNs, (i_1, i_2, \dots, i_k) traverses all the k -tuple combination of $(1, 2, \dots, n)$ and $C_n^k = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

Based on the interaction operational rules of the IFNs, we can derive the aggregation result from Definition 7 shown as theorem 5.

Theorem 5. Suppose $\tilde{b}_i = (u_i, v_i) (i = 1, 2, \dots, n)$ is a collection of IFNs, and $k = 1, 2, \dots, n$, then, the aggregated value from (23) is still an IFN, and

$$IFWIMSM^{(k)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) =$$

$$\left(\left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (1 - u_{i_j})^{w_{i_j}} + (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right) + \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}},$$

$$1 - \left[1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (1 - u_{i_j})^{w_{ij}} + (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) + \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right)^{\frac{1}{C_n^k}} \right]^{\frac{1}{k}}. \tag{24}$$

Proof.

Firstly, we can calculate $w_{i_j} \tilde{b}_{i_j}$, and get

$$w_{i_j} \tilde{b}_{i_j} = \left(1 - (1 - u_{i_j})^{w_{ij}}, (1 - u_{i_j})^{w_{ij}} - (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right),$$

and

$$\begin{aligned} \otimes_{j=1}^k w_{i_j} \tilde{b}_{i_j} &= \left(\prod_{j=1}^k \left(1 - (1 - u_{i_j})^{w_{ij}} + (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) - \prod_{j=1}^k \left((1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right), \right. \\ &\left. 1 - \prod_{j=1}^k \left(1 - (1 - u_{i_j})^{w_{ij}} + (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) \right). \end{aligned}$$

Then we have

$$\begin{aligned} \oplus_{1 \leq i_1 < \dots < i_k \leq n} \left(\otimes_{j=1}^k w_{i_j} \tilde{b}_{i_j} \right) &= \left(1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (1 - u_{i_j})^{w_{ij}} + (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) \right) + \right. \\ &\left. \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right), \\ &\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (1 - u_{i_j})^{w_{ij}} + (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) + \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) - \\ &\left. \prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right), \end{aligned}$$

and then we get

$$\begin{aligned} \frac{1}{C_n^k} \left(\oplus_{1 \leq i_1 < \dots < i_k \leq n} \left(\otimes_{j=1}^k w_{i_j} \tilde{b}_{i_j} \right) \right) &= \\ \left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (1 - u_{i_j})^{w_{ij}} + (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) + \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) \right)^{C_n^k} \right), \end{aligned}$$

$$\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (1 - u_{i_j})^{w_{i_j}} + (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) + \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{C_n^k} - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^k}} \right).$$

Finally, we obtain

$$IFWIMSM^{(k)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \left(\frac{\bigoplus_{\substack{1 \leq i_1 < \dots < i_k \leq n \\ < i_k \leq n}} \left(\bigotimes_{j=1}^k (w_{i_j} \tilde{b}_{i_j}) \right) \right)^{1/k}}{C_n^k} = \left(\left(1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - (1 - u_{i_j})^{w_{i_j}} + (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) + \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^k}} \right)^{1/k} - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^k}} \right)^{1/k} - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^k}} + \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^k}} \right)^{1/k} \right).$$

In addition, it is also an IFN.

It is easy to prove that the IFWIMSM operator has the following properties.

Theorem 6 (Idempotency). Let $\tilde{b}_j = (u_j, v_j) (j=1, 2, \dots, n)$ be a collection of IFNs, if $\tilde{b}_j = \tilde{b} = (u, v), j=1, 2, \dots, n$, then

$$IFWIMSM = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \tilde{b} = (u, v). \tag{25}$$

The proof of this theorem is similar to Theorem 3.

Theorem 7 (Commutativity). Suppose $\tilde{b}_j = (u_j, v_j) (j=1, 2, \dots, n)$ is a collection of IFNs, and $\tilde{b}'_j = (u'_j, v'_j)$ is any permutation of \tilde{b}_j , then

$$IFWIMSM^{(k)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = IFWIMSM^{(k)}(\tilde{b}'_1, \tilde{b}'_2, \dots, \tilde{b}'_n). \tag{26}$$

The proof of this theorem is similar to Theorem 4.

As well, we can discuss some special cases of the IFWIMSM operator with different values of the parameter k .

(1) If $k = 1$, then the IFWIMSM operator (24) will reduce to the following formula:

$$\begin{aligned}
 IFWIMSM^{(1)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = & \left(\left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^1 \left(1 - \left((1 - u_{i_j})^{w_{i_j}} - (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right) + \prod_{j=1}^1 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right) \right)^{\frac{1}{C_n^1}} + \right. \\
 & \left. \left(\prod_{1 \leq i_1 \leq n} \prod_{j=1}^1 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}} - \left(\left(\prod_{1 \leq i_1 \leq n} \prod_{j=1}^1 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}} \\
 & 1 - \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^1 \left(1 - (1 - u_{i_j})^{w_{i_j}} + (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right) + \prod_{j=1}^1 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right)^{\frac{1}{C_n^1}} \\
 & \left. \left(\prod_{1 \leq i_1 \leq n} \prod_{j=1}^1 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}} = \\
 & \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left((1 - u_{i_j})^{w_{i_j}} \right) \right)^{\frac{1}{n}}, \left(\prod_{1 \leq i_1 \leq n} (1 - u_{i_j})^{w_{i_j}} \right)^{\frac{1}{n}} - \left(\prod_{1 \leq i_1 \leq n} (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{n}} \right) \text{ (let } i_j = j) = \\
 & \left(1 - \left(\prod_{j=1}^n (1 - u_j)^{w_j} \right)^{\frac{1}{n}}, \left(\prod_{j=1}^n (1 - u_j)^{w_j} \right)^{\frac{1}{n}} - \left(\prod_{j=1}^n (1 - (u_j + v_j)^{w_j} \right)^{\frac{1}{n}} \right). \tag{27}
 \end{aligned}$$

(2) If $k = 2$, then the IFWIMSM operator (24) will reduce to the intuitionistic fuzzy weighted interaction BM (IFWIBM) ($p = q = 1$) operator. It is shown as follows:

$$\begin{aligned}
 IFWIMSM^{(2)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = & \left(\left(1 - \left(\prod_{1 \leq i_1 < i_2 \leq n} \left(1 - \prod_{j=1}^2 \left(1 - \left((1 - u_{i_j})^{w_{i_j}} - (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right) + \prod_{j=1}^2 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right) \right)^{\frac{1}{C_n^2}} + \right. \\
 & \left. \left(\prod_{1 \leq i_1 < i_2 \leq n} \prod_{j=1}^2 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^2}} \right)^{\frac{1}{2}} - \left(\left(\prod_{1 \leq i_1 < i_2 \leq n} \prod_{j=1}^2 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^2}} \right)^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 & 1 - \left[1 - \left(\prod_{1 \leq i_1 < i_2 \leq n} \left(1 - \prod_{j=1}^2 \left(1 - (1 - u_{i_j})^{w_{i_j}} + (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) + \prod_{j=1}^2 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right)^{\frac{1}{C_n^2}} + \right. \\
 & \left. \left(\prod_{1 \leq i_1 < i_2 \leq n} \prod_{j=1}^2 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^2}} \right)^{\frac{1}{2}} = \\
 & \left(\left(1 - \left(\prod_{1 \leq i_1 < i_2 \leq n} \left(1 - \prod_{j=1}^2 \left(1 - (1 - u_{i_j})^{w_{i_j}} + (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) + \prod_{j=1}^2 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right) \right)^{\frac{2}{n(n-1)}} + \right. \\
 & \left. \left(\prod_{1 \leq i_1 < i_2 \leq n} \prod_{j=1}^2 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}} - \left(\left(\prod_{1 \leq i_1 < i_2 \leq n} \prod_{j=1}^2 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}}, \\
 & 1 - \left[1 - \left(\prod_{1 \leq i_1 < i_2 \leq n} \left(1 - \prod_{j=1}^2 \left(1 - (1 - u_{i_j})^{w_{i_j}} + (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) + \prod_{j=1}^2 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right) \right]^{\frac{2}{n(n-1)}} + \\
 & \left. \left(\prod_{1 \leq i_1 < i_2 \leq n} \prod_{j=1}^2 (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}} = WIFIBM^{1,1}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n). \tag{28}
 \end{aligned}$$

(3) If $k = n$, then the IFWIMSM operator (24) will reduce to the following formula:

$$\begin{aligned}
 & IFWIMSM^{(n)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \\
 & \left(\left(1 - \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \left(1 - \prod_{j=1}^n \left(1 - (1 - u_{i_j})^{w_{i_j}} + (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) + \prod_{j=1}^n (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right) \right)^{\frac{1}{C_n^n}} + \right. \\
 & \left. \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^n}} \right)^{\frac{1}{n}} - \left(\left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^n}} \right)^{\frac{1}{n}}, \\
 & 1 - \left[1 - \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \left(1 - \prod_{j=1}^n \left(1 - (1 - u_{i_j})^{w_{i_j}} + (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) + \prod_{j=1}^n (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right) \right) \right]^{\frac{1}{C_n^n}} + \\
 & \left. \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n (1 - (u_{i_j} + v_{i_j}))^{w_{i_j}} \right)^{\frac{1}{C_n^n}} \right)^{\frac{1}{n}}
 \end{aligned}$$

$$\begin{aligned}
 & \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n \left(1 - (u_{i_j} + v_{i_j}) \right)^{w_{ij}} \right)^{\frac{1}{C_n^n}} \Bigg)^{\frac{1}{n}} = \\
 & \left(1 - \prod_{1 \leq i_1 < \dots < i_n \leq n} \left(1 - \prod_{j=1}^n \left(1 - (1 - u_{i_j})^{w_{ij}} + (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) + \prod_{j=1}^n \left(1 - (u_{i_j} + v_{i_j}) \right)^{w_{ij}} \right) \right) + \\
 & \prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n \left(1 - (u_{i_j} + v_{i_j}) \right)^{w_{ij}} \Bigg)^{\frac{1}{n}} - \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n \left(1 - (u_{i_j} + v_{i_j}) \right)^{w_{ij}} \right)^{\frac{1}{n}}, \\
 & 1 - \left(1 - \prod_{1 \leq i_1 < \dots < i_n \leq n} \left(1 - \prod_{j=1}^n \left(1 - (1 - u_{i_j})^{w_{ij}} + (1 - (u_{i_j} + v_{i_j}))^{w_{ij}} \right) + \prod_{j=1}^n \left(1 - (u_{i_j} + v_{i_j}) \right)^{w_{ij}} \right) \right) + \\
 & \prod_{1 \leq i_1 < \dots < i_n \leq n} \prod_{j=1}^n \left(1 - (u_{i_j} + v_{i_j}) \right)^{w_{ij}} \Bigg)^{\frac{1}{n}}. \tag{29}
 \end{aligned}$$

3. A group decision-making approach based on the IFWIMSM operator

In this section, we will apply the IFIWMSM operator to solve the MAGDM problems.

3.1. Description of the MAGDM problems

A MAGDM problem with IFNs is described as follows. Suppose $\{A_1, A_2, \dots, A_m\}$ is the set of alternatives and $\{C_1, C_2, \dots, C_n\}$ is the set of attributes which weight vector is $w = (w_1, w_2, \dots, w_n)^T$ with $w_j \geq 0, j = 1, 2, \dots, n$, and $\sum_{j=1}^n w_j = 1$. Further, suppose $\{D_1, D_2, \dots, D_t\}$ is the set of decision makers (DMs) and $\omega = (\omega_1, \omega_2, \dots, \omega_t)$ be the weight vector of them with $\omega_k \geq 0 (k = 1, 2, \dots, t)$, and $\sum_{k=1}^t \omega_k = 1$. Let $\tilde{R}^k = [\tilde{r}_{ij}^k]_{m \times n}$ be the decision matrix of MAGDM problems, where $\tilde{r}_{ij}^k = (u_{ij}^k, v_{ij}^k)$ is the evaluation information expressed by the IFN with respect to alternative A_i for attribute C_j given by the DM D_k . Then, the goal of this decision problem is to rank alternatives.

Based on the IFWIMSM operator proposed in section 2, we will give its application in the MAGDM problems and establish the detailed decision-making process shown as follows.

3.2. The decision-making steps based on the IFWIMSM operator

Step 1. Normalize the attribute values.

In real decision-making, the attribute values have two types, i.e., cost attribute and benefit attribute. In order to eliminate the difference in types, we need convert them to the same

type. In general, we need convert cost type to benefit type. If the attribute value $\tilde{r}_{ij}^k = (u_{ij}^k, v_{ij}^k)$ is cost type, it can be transformed to benefit one shown as follows (suppose the transformed attribute value is still expressed by \tilde{r}_{ij}^k):

$$\tilde{r}_{ij}^k = (v_{ij}^k, u_{ij}^k). \tag{30}$$

Step 2. Aggregate the evaluation information of individual DM to collective information by IFWIMSM operator shown as follows:

$$\tilde{r}_{ij} = IFWIMSM^{(k)}(\tilde{r}_{ij}^1, \tilde{r}_{ij}^2, \dots, \tilde{r}_{ij}^t). \tag{31}$$

Step 3. Aggregate the evaluation information of each attribute to the comprehensive evaluation value of each alternative by IFWIMSM operator shown as follows:

$$\tilde{z}_i = IFWIMSM^{(k)}(\tilde{r}_{i1}, \tilde{r}_{i2}, \dots, \tilde{r}_{in}). \tag{32}$$

Step 4. Calculate the score function $S(\tilde{z}_i) (i=1,2,\dots,m)$ of the collective overall values $\tilde{z}_i (i=1,2,\dots,m)$, and then rank all the alternatives $\{A_1, A_2, \dots, A_m\}$. When two score functions $S(\tilde{z}_i)$ and $S(\tilde{z}_j)$ are equal, it is necessary to calculate their accuracy functions $H(\tilde{z}_i)$ and $H(\tilde{z}_j)$, then we can rank them by accuracy functions.

Step 5. Rank the alternatives.

Rank all the alternatives $\{A_1, A_2, \dots, A_m\}$ and choose the best one(s) according to score function $S(\tilde{z}_i)$ and accuracy function $H(\tilde{z}_i)$.

Step 6. End.

4. An illustrative example

In order to show the application of the proposed method, we will give an example about the route selection for five possible distribution schemes $\{A_1, A_2, A_3, A_4, A_5\}$. There are three DMs $D_k (k=1,2,3)$ to evaluate these five alternatives according to four attributes which are shown as follows: the risk analysis (C_1), the cost of transportation analysis (C_2), the convenience of transportation analysis (C_3) and the environmental impact analysis (C_4). The evaluation results are used to construct three decision matrices $\tilde{R}^k = [\tilde{r}_{ij}^k]_{5 \times 4} (k=1,2,3)$ listed in Tables 1–3, where \tilde{r}_{ij}^k can be expressed as IFN (u_{ij}^k, v_{ij}^k) . Suppose the weight vector of three DMs is $\omega = (0.35, 0.40, 0.25)^T$ and the weight vector of the attributes is $w = (0.2, 0.1, 0.3, 0.4)^T$. The goal of this MAGDM problem is to select the optimization route for distribution schemes.

Table 1. Intuitionistic fuzzy decision matrix \tilde{R}^1 given by D_1

	C_1	C_2	C_3	C_4
A_1	(0.5,0.4)	(0.6,0.3)	(0.3,0.6)	(0.5,0.4)
A_2	(0.6,0.3)	(0.6,0.3)	(0.6,0.2)	(0.6,0.3)
A_3	(0.5,0.4)	(0.2,0.6)	(0.6,0.2)	(0.4,0.4)
A_4	(0.6,0.2)	(0.7,0.2)	(0.5,0.4)	(0.4,0.4)
A_5	(0.4,0.3)	(0.7,0.2)	(0.4,0.5)	(0.4,0.5)

Table 2. Intuitionistic fuzzy decision matrix \tilde{R}^2 given by D_2

	C_1	C_2	C_3	C_4
A_1	(0.4,0.2)	(0.6,0.2)	(0.5,0.4)	(0.5,0.3)
A_2	(0.5,0.3)	(0.6,0.2)	(0.6,0.2)	(0.5,0.4)
A_3	(0.4,0.4)	(0.3,0.5)	(0.5,0.4)	(0.4,0.2)
A_4	(0.5,0.4)	(0.7,0.2)	(0.5,0.2)	(0.7,0.2)
A_5	(0.6,0.3)	(0.7,0.2)	(0.4,0.2)	(0.6,0.2)

Table 3. Intuitionistic fuzzy decision matrix \tilde{R}^3 given by D_3

	C_1	C_2	C_3	C_4
A_1	(0.4,0.5)	(0.5,0.2)	(0.5,0.3)	(0.5,0.2)
A_2	(0.5,0.4)	(0.5,0.3)	(0.6,0.2)	(0.7,0.2)
A_3	(0.4,0.5)	(0.4,0.4)	(0.5,0.3)	(0.6,0.3)
A_4	(0.5,0.3)	(0.4,0.5)	(0.5,0.4)	(0.5,0.3)
A_5	(0.6,0.2)	(0.5,0.3)	(0.4,0.4)	(0.6,0.3)

4.1. The decision-making steps

To get the best alternative(s), the steps are shown as following.

Step 1. Normalize the attribute values.

All the measured values are the same type, so they do not need to do the normalization.

Step 2. Aggregate the evaluation information of individual DM to collective information by IFIWMSM operator (without loss of generality, we suppose $k = 3$).

$$\begin{aligned} \tilde{r}_{11} &= (0.176, 0.265), \tilde{r}_{12} = (0.256, 0.186), \tilde{r}_{13} = (0.177, 0.331), \tilde{r}_{14} = (0.212, 0.230), \\ \tilde{r}_{21} &= (0.227, 0.264), \tilde{r}_{22} = (0.253, 0.208), \tilde{r}_{23} = (0.265, 0.150), \tilde{r}_{24} = (0.260, 0.276), \\ \tilde{r}_{31} &= (0.175, 0.315), \tilde{r}_{32} = (0.109, 0.306), \tilde{r}_{33} = (0.232, 0.235), \tilde{r}_{34} = (0.185, 0.210), \\ \tilde{r}_{41} &= (0.232, 0.235), \tilde{r}_{42} = (0.263, 0.234), \tilde{r}_{43} = (0.206, 0.257), \tilde{r}_{44} = (0.246, 0.221), \\ \tilde{r}_{51} &= (0.237, 0.204), \tilde{r}_{52} = (0.308, 0.201), \tilde{r}_{53} = (0.159, 0.250), \tilde{r}_{54} = (0.221, 0.270). \end{aligned}$$

Step 3. Aggregate the evaluation information of each attribute to the comprehensive evaluation value of each alternative by IFWIMSM operator (without loss of generality, we let $k = 3$).

$$\begin{aligned} \tilde{z}_1 &= (0.0530, 0.7901), \tilde{z}_2 = (0.0679, 0.7821), \tilde{z}_3 = (0.0483, 0.7885), \tilde{z}_4 = (0.0634, 0.7857), \\ \tilde{z}_5 &= (0.0613, 0.7837). \end{aligned}$$

Step 4. Calculate the score function $S(\tilde{z}_i) (i=1,2,\dots,4)$ of the collective overall values $\tilde{z}_i (i=1,2,\dots,5)$.

$$S(\tilde{z}_1) = -0.7371, S(\tilde{z}_2) = -0.7141, S(\tilde{z}_3) = -0.7402, S(\tilde{z}_4) = -0.7223, S(\tilde{z}_5) = -0.7224.$$

Step 5. Rank the alternatives.

On the basis of the score functions $S(\tilde{z}_i) (i=1,2,3,4)$, we can rank the alternatives $\{A_1, A_2, A_3, A_4, A_5\}$ shown as follows.

$$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3.$$

So, the best alternative is A_2 .

Step 6. End.

4.2. The influence of the parameters k on decision-making result of this example

In order to clarify the influence of the parameter k on decision-making of this example, we use the different values k in steps 2 and 3 to rank the alternatives. The ranking results are shown in Table 4.

Table 4. Ordering of the alternatives by utilizing the different k

k	score function $S(\tilde{z}_i)$	Ranking
$k = 1$	$S(\tilde{z}_1) = -0.0383, S(\tilde{z}_2) = -0.0125$ $S(\tilde{z}_3) = -0.0329, S(\tilde{z}_4) = -0.019$ $S(\tilde{z}_5) = -0.0251$	$A_2 \succ A_4 \succ A_5 \succ A_3 \succ A_1$
$k = 2$	$S(\tilde{z}_1) = -0.7337, S(\tilde{z}_2) = -0.7075$ $S(\tilde{z}_3) = -0.7332, S(\tilde{z}_4) = -0.7178$ $S(\tilde{z}_5) = -0.7211$	$A_2 \succ A_4 \succ A_5 \succ A_3 \succ A_1$
$k = 3$	$S(\tilde{z}_1) = -0.7371, S(\tilde{z}_2) = -0.7141$ $S(\tilde{z}_3) = -0.7402, S(\tilde{z}_4) = -0.7223$ $S(\tilde{z}_5) = -0.7224$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$

As we can see from Table 4, the ranking results with the different parameter k are slightly different, but the best choice is always A_2 . In addition, we can also find that the score functions on the basis of the IFWIMSM operator became smaller as the parameter k increases, so we can regard parameter k as the DM’s risk preference. With the increase of the parameter k , the decision-making results will be changed from the optimism to pessimism. In real practical decision-making situations, DMs can choose the appropriate value in accordance with their risk preferences.

4.3. The verification of the validity

To prove the effectiveness of the developed method in this paper, we solve the same illustrative example by using the three existing MAGDM methods including the weighted intuitionistic fuzzy Maclaurin symmetric mean (WIFMSM) operator proposed by Qin and Liu (2014), the intuitionistic fuzzy weighted average (IFWA) operator proposed by Xu (2007).

For convenience, we let $p = q = 1, k = 3$, then the final ranking orders of the alternatives obtained by the above three methods are described in Table 5.

Table 5. Ranking results by different method

Method	Aggregation operator	Score values $S(Z_i)$	Ranking
Xu (2007)	IFWA	$S(\tilde{z}_1) = 0.1489, S(\tilde{z}_2) = 0.3164,$ $S(\tilde{z}_3) = 0.1377, S(\tilde{z}_4) = 0.2586,$ $S(\tilde{z}_5) = 0.2211$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
Qin and Liu (2014)	WIFMSM	$S(\tilde{z}_1) = -0.3352, S(\tilde{z}_2) = -0.2678,$ $S(\tilde{z}_3) = -0.3704, S(\tilde{z}_4) = -0.2864,$ $S(\tilde{z}_5) = -0.2944$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
Proposed method	IFWIMSM	$S(\tilde{z}_1) = -0.7371, S(\tilde{z}_2) = -0.7141$ $S(\tilde{z}_3) = -0.7402, S(\tilde{z}_4) = -0.7223$ $S(\tilde{z}_5) = -0.7224$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$

From Table 5, we can find that there are the same ranking results by using three methods. So the method in this paper is effective and feasible.

4.4. Further compared with other methods

From the above subsections, we have testified the validity of our proposed method. However, we find that they have the same ranking results, so it is difficult to illustrate the advantages of our method and the drawbacks of the existing methods in some situations. So we give two examples to show that our method is more extensive. In example 4, we will show the advantages of MSM operator of IFNs comparing with BM operator of IFNs, and example 5 will illustrate the advantages of interaction operational rules of IFNs comparing with the traditional rules.

Example 4. We solve the same illustrative example by the MAGDM methods based on the weighted intuitionistic fuzzy interaction BM (WIFIBM) operator proposed by Y. D. He, Z. He, and Chen (2015) and the proposed method in this paper because these two methods adopt the same operational rules of IFNs. For easily comparing, we let $k = 2$ and $k = 3$. Then we can get the results shown in Table 6.

From table 6, we can see that the ranking results based on WIFIBM operator ($p = q = 1$) and IFWIMSM operator ($k = 2$) are same. Obviously, this conclusion can be easily explained that these two methods consider interrelationship only for two attributes, even, we can also know that when $k = 2$, the IFWIMSM operator will reduce to the WIFIBM ($p = q = 1$). However, when $k = 3$, the method based on IFWIMSM operator can consider interrelationship for three attributes, and the ranking result is different from the method proposed by He et al. (2015) because it is only for two attributes. In addition, we also know that the WIFIBM operator proposed by He et al. (2015) is a special case of the IFWIMSM operator proposed in this paper. So the IFWIMSM is more flexible than WIFIBM operator.

Table 6. Ranking results by different methods and parameters

Method and operators parameters	Score values $S(Z_i)$	Ranking
Method by He et al. $p = q = 1$ (2015) based on WIFIBM	$S(\tilde{z}_1) = -0.0423, S(\tilde{z}_2) = -0.0191$ $S(\tilde{z}_3) = -0.0372, S(\tilde{z}_4) = -0.0257$ $S(\tilde{z}_5) = -0.0311$	$A_2 \succ A_4 \succ A_5 \succ A_3 \succ A_1$
Our proposed method $k = 2$ Based on IFWIMSM	$S(\tilde{z}_1) = -0.7337, S(\tilde{z}_2) = -0.7075$ $S(\tilde{z}_3) = -0.7332, S(\tilde{z}_4) = -0.7178$ $S(\tilde{z}_5) = -0.7211$	$A_2 \succ A_4 \succ A_5 \succ A_3 \succ A_1$
Our proposed method $k = 3$ Based on IFWIMSM	$S(\tilde{z}_1) = -0.7371, S(\tilde{z}_2) = -0.7141$ $S(\tilde{z}_3) = -0.7402, S(\tilde{z}_4) = -0.7223$ $S(\tilde{z}_5) = -0.7224$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$

Example 5. Assume that a company wants to choose a new salary plan and there are five choices $\{A_1, A_2, A_3, A_4, A_5\}$, and four attributes (let their weight vector be $w = (0.25, 0.25, 0.25, 0.25)^T$) are shown as follows: the employee satisfaction (C_1), the feasibility analysis (C_2), the influence of surrounding environment (C_3) and the influence of social-politic (C_4).

The DM D evaluates the plans A_i ($i = 1, 2, 3, 4, 5$) with respect to the attributes C_j ($j = 1, 2, 3, 4$) by the IFNs and the decision matrix \tilde{R} is listed in Table 7, where \tilde{r}_{ij} can be expressed as (u_{ij}, v_{ij}) .

Table 7. Intuitionistic fuzzy decision matrix \tilde{R} given by D

	C_1	C_2	C_3	C_4
A_1	(0.65, 0.25)	(0.71, 0.23)	(0.53, 0.32)	(0.62, 0)
A_2	(0.62, 0.33)	(0.64, 0.30)	(0.56, 0.34)	(0.68, 0.2)
A_3	(0.65, 0.32)	(0.53, 0)	(0.57, 0.34)	(0.71, 0.25)
A_4	(0.66, 0.23)	(0.77, 0.18)	(0.48, 0.24)	(0.57, 0.37)
A_5	(0.58, 0.23)	(0.62, 0.24)	(0.72, 0.12)	(0.64, 0.15)

The aggregation results for the different methods are shown in Table 8.

From Table 8, we obtain that the best choice based on the IFWA operator proposed by Xu (2007) is A_1 and it is same with WIFMSM operator (Qin & Liu, 2014). While the best choices produced by the WIFIBM operator proposed by He et al. (2015) and IFWIMSM operator proposed in this paper are both A_5 . Because the methods proposed by Xu (2007) and by Qin and Liu (2014) are all based on traditional operations which do not consider the interactions between membership function and non-membership function of different IFNs, and there exist some weaknesses in operations of IFNs when one of non-memberships is zero, it is possible to produce the unreasonable results. In this example, because non-memberships in the attributes \tilde{r}_{14} and \tilde{r}_{32} are zero, the aggregation results for the choices A_1 and A_3 are

unreasonable, so the ranking results for five choices are not reasonable. However, the methods proposed by He et al. (2015) and in this paper can consider the interactions between membership function and non-membership function of different IFNs, and they can relieve the weakness when the non-membership function of any one IFN is zero. So the ranking results are reasonable, i.e., A_5 is the best choice and not A_1 .

Table 8. Ranking results by different methods

Method and adopted aggregation Operators	Score values $S(Z_i)$	Ranking
Method by Xu (2007) based on IFWA operator	$S(\tilde{z}_1) = 0.6331, S(\tilde{z}_2) = 0.3410$ $S(\tilde{z}_3) = 0.6216, S(\tilde{z}_4) = 0.3901$ $S(\tilde{z}_5) = 0.4663$	$A_1 \succ A_3 \succ A_5 \succ A_4 \succ A_2$
Method by Qin and Liu (2014) based on WIFMSM operator	$S(\tilde{z}_1) = -0.3774, S(\tilde{z}_2) = -0.5166$ $S(\tilde{z}_3) = -0.4077, S(\tilde{z}_4) = -0.4923$ $S(\tilde{z}_5) = -0.4271$	$A_1 \succ A_3 \succ A_5 \succ A_4 \succ A_2$
Method by He et al. (2015) based on WIFIBM ($p = q = 1$) operator	$S(\tilde{z}_1) = 0.9868, S(\tilde{z}_2) = 0.9896$ $S(\tilde{z}_3) = 0.9839, S(\tilde{z}_4) = 0.9865$ $S(\tilde{z}_5) = 0.9902$	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$
Our proposed method based on IFWIMSM ($k = 3$) operator	$S(\tilde{z}_1) = -0.5022, S(\tilde{z}_2) = -0.5421$ $S(\tilde{z}_3) = -0.5032, S(\tilde{z}_4) = -0.5203$ $S(\tilde{z}_5) = -0.5002$	$A_5 \succ A_1 \succ A_3 \succ A_4 \succ A_2$

In addition, although the best choice for the methods proposed by He et al. (2015) and in this paper is the same, the ranking for these two methods are different. The reason is that the method proposed by He et al. (2015) is based on the interrelationship only for two attributes and the method in this paper is based on interrelationship for three attributes.

In a word, our method can overcome the weakness of the some existing methods proposed by Xu (2007), Qin and Liu (2014) which are based on the traditional operational rules. In addition, our method is more general because it can consider interrelationship from two attributes to n attributes.

In the following, we will give some comparisons of the three methods and our proposed method, which are listed in Table 9.

According to the above analysis, the comparisons between our proposed method and the existing three methods can be described as follows.

(1) Compared with the method based on the IFWA operator, we can find that the method proposed by Xu (2007) can describe fuzzy information easier. However, this method is based on the assumption that the attributes are independent and it doesn't consider the interrelationship between them. The improved operator in this paper not only considers the interrelationship between two attributes but also can take interrelationship among multi-attributes into account.

Table 9. The comparisons of different methods

Methods	whether captures interrelationship of two attributes	whether captures interrelationship of multiple attributes	whether considerate the interaction between the membership function and non-membership function
Xu (2007)	No	No	No
Qin and Liu (2014)	Yes	Yes	No
He et al. (2015)	Yes	No	Yes
Proposed method	Yes	Yes	Yes

Moreover, in aggregation operations, the method proposed by Xu (2007) uses the traditional operational rules, while our method makes use of the interaction operational rules. As we all know, the traditional operations do not consider the interactions between membership function and non-membership function of different IFNs, so they can get unreasonable results in some special cases, especially when one of the non-memberships is zero. However, the interaction operational rules can consider the interactions between membership function and non-membership function sufficiently, so the method proposed in this paper is more reasonable to produce the ranking result because it can overcome the weakness when one of the non-memberships is zero.

(2) Compared with the method proposed by Qin and Liu (2014) based on the WIFMSM operator, obviously, these two methods are all based on MSM operator which can consider the interrelationship among multiple attributes. However, the method proposed by Qin and Liu (2014) is based on the traditional operational rules which do not consider the interactions between membership function and non-membership function of different IFSs, and it can produce the unreasonable result when one of the non-memberships is zero, while the method proposed in this paper can take account of the interactions between membership function and non-membership function, and overcomes some existing problems by the traditional operational rules.

(3) Compared with the method proposed by He et al. (2015) based on the WIFIBM operator. Obviously, these two methods are all based on the interaction operational rules of IFNs which can overcome the existing problems when one of the non-memberships is zero. However, the method proposed by He et al. (2015) adopts the BM operator which can only consider the interrelationship between two attributes while the method proposed in this paper not only considers the interrelationship between two attributes but also can take interrelationship among multi-attributes into account. In addition, we can also know that the WIFIBM operator is a special case of the WIFMSM operator when $k = 2$. Obviously, the method based on the WIFMSM operator is more general than the method based on the WIFIBM operator.

According to the comparisons and analysis above, the IFWIMSM operator developed in this paper is better than the existing other methods for aggregating the IFNs. Therefore, the IFWIMSM operator is more suitable to deal with the problem of MAGDM in the intuitionistic fuzzy environment.

Conclusions

In this paper, we extended the MSM operator to IFNs based on the interaction operations for the IFNs and proposed some interaction MSM operators for IFNs, such as the IFIMSM and IFWIMSM operators, then we discussed some desirable characteristics of them, such as idempotency and commutativity. Further we analyzed some special cases of these operators, and proposed a method for the MAGDM problems based on the IFWIMSM operator. Comparing with the existing methods, the proposed method is more general than some existing methods. The significant advantages are that they can capture the interrelationship among the multi-input arguments which have the flexibility by the parameters k , and they can also consider interactions between membership function and non-membership function of IFNs which can overcome some existing problems when one of the non-memberships is zero.

In further research, it is necessary and significant to take the applications of these operators to solve the real decision-making problems, such as evaluations on population resources and environment (Zha & Kavuri, 2016; Zhang, He & Pan, 2017; Zhang & Shi, 2016; Zhu, 2017) or Chinese culture (Hou, 2016), or the proposed operators are extended to some new fuzzy information (Liu & Chen, 2018; P. D. Liu, Zhang, X. Liu, & Wang, 2016).

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