TECHNOLOGICAL AND ECONOMIC DEVELOPMENT OF ECONOMY
Baltic Journal on Sustainability
2009
15(2): 310–325

GLOBAL OPTIMIZATION USING THE BRANCH-AND-BOUND ALGORITHM WITH A COMBINATION OF LIPSCHITZ BOUNDS OVER SIMPLICES

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Received 16 September 2008; accepted 4 May 2009

Abstract. Many problems in economy may be formulated as global optimization problems. Most numerically promising methods for solution of multivariate unconstrained Lipschitz optimization problems of dimension greater than 2 use rectangular or simplicial branch-and-bound techniques with computationally cheap, but rather crude lower bounds. The proposed branch-and-bound algorithm with simplicial partitions for global optimization uses a combination of 2 types of Lipschitz bounds. One is an improved Lipschitz bound with the first norm. The other is a combination of simple bounds with different norms. The efficiency of the proposed global optimization algorithm is evaluated experimentally and compared with the results of other well-known algorithms. The proposed algorithm often outperforms the comparable branch-and-bound algorithms.

Keywords: branch-and-bound algorithm, Lipschitz optimization, global optimization, Lipschitz bound.

Reference to this paper should be made as follows: Paulavičius, R.; Žilinskas, J. 2009. Global optimization using the branch-and-bound algorithm with a combination of Lipschitz bounds over simplices, *Technological and Economic Development of Economy* 15(2): 310–325.

1. Introduction

It is often necessary to optimize an objective function for economic problems. For example, estimates of the parameter values of the models of markets in disequilibrium may be found by maximizing likelihood functions. The models of markets in disequilibrium are used to model markets, characterized by excess supplies or demands (Maddala 1983), which means that supply and demand in these models may be not equal. In Model 1 of markets in disequilibrium from (Maddala and Nelson 1974), the actual quantity transacted during some period

ISSN 1392-8619 print/ISSN 1822-3613 online doi: 10.3846/1392-8619.2009.15.310-325
http://www.tede.vgtu.lt

is observed and it is not known whether the market was in equilibrium, in excess demand, or in excess supply during this period. The model consists of the equations:

$$D_t = \alpha_D + X'_{Dt}\beta_D + u_{Dt},$$

$$S_t = \alpha_S + X'_{St}\beta_S + u_{St},$$

$$Q_t = \min(D_t, S_t),$$

where D_t, S_t, and Q_t are the quantities demanded, supplied, and transacted during the period t, respectively, α_D is a demand constant parameter, α_S – a supply constant parameter, X_{Dt} – a vector of observed variables that influence demand during the period t, X_{St} – a vector of observed variables that influence supply during the period t, β_D – a vector of demand parameters (the length of β_D is the same as that of X_{Dt}), β_S – a vector of supply parameters (the length of β_S is the same as that of X_{St}), u_{Dt} and u_{St} are random residuals.

It is assumed that u_{Dt} and u_{St} are independently and normally distributed with zero mean and variances σ_D^2 and σ_S^2, respectively. The density of Q_t is derived in (Maddala and Nelson 1974). The estimates of the model parameter values α_D, β_D, α_S, β_S, σ_D^2, and σ_S^2 may be found by maximizing the likelihood function

$$L = \prod_t G_t,$$

where

$$G_t = f_{Dt} \cdot F_{St} + f_{St} \cdot F_{Dt},$$

or the log likelihood function

$$L = \sum_t \log(G_t).$$

Maddala and Nelson (1974) have tried to estimate parameters of the model with data from housing starts (Fair 1971) by local search techniques and concluded that local searches converged to different values, suggesting the existence of multiple maxima. Dorsey and Mayer (1995) have tried to optimize the problem by adaptive random search, simulated annealing, and genetic algorithms, but none of them converged to the best solution found so far. Jerrell and Campione (2001) have tried to optimize the problem by random search techniques (genetic algorithm, evolution strategy, and simulated annealing) and have found that random search methods do not terminate at the same point more than once. The possibility of estimating bounds for this econometric likelihood function by balanced random interval arithmetic is experimentally investigated in (Žilinskas and Bogle 2006). It can be concluded that the problem is a difficult global optimization problem. In this paper, an improved general algorithm for multidimensional Lipschitz global optimization is proposed and experimentally investigated.

2. Lipschitz optimization

Consider the problem of global maximization or minimization of a Lipschitz continuous objective function $f : \Re^n \to \Re$ over a given compact subset $D \subseteq \Re^n$. Since minimization can be transformed into maximization by changing the sign of the objective function, we will consider only the maximization problem

$$f^* = \max_{x \in D} f(x). \tag{1}$$

Apart from the global optimum f^*, one or all global optimizers $x^*: f(x^*) = f^*$ should be found. In this paper Lipschitz optimization is considered. A function $f: D \to \Re$, $D \subseteq \Re^n$ is said to be Lipschitz, if it satisfies the condition

$$|f(x) - f(y)| \leq L\|x - y\|, \forall x, y \in D, \tag{2}$$

where $L > 0$ is a constant called the Lipschitz constant and $\|\cdot\|$ denotes the norm. The Euclidean norm is most often used in the Lipschitz optimization, but other norms can also be considered.

The most studied case of problem (1) is the unconstrained univariate one ($n = 1$), for which numerous algorithms have been proposed, compared, and theoretically investigated. An excellent comprehensive survey is contained in (Hansen and Jaumard 1995). In the present paper, we are mainly interested in the multivariate case ($n \geq 2$).

In the Lipschitz optimization the upper bound of the optimal value f^* is evaluated by exploiting Lipschitz condition. It follows from (2) that, for all $x, y \in D$

$$f(x) \leq f(y) + L\|x - y\|.$$

If $y \in D$ is fixed, then the concave function

$$F(x, y) = f(y) + L\|x - y\| \tag{3}$$

overestimates $f(x)$ over D. Let T be a finite set of distinct points in D. Then, the sharpest upper bound over D, given the function values $f(y), y \in T$, and the Lipschitz constant L, is provided by

$$\max_{x \in D} \min_{y \in T} F(x, y). \tag{4}$$

In the univariate case, the function F is piecewise linear, and (4) can be determined in a simple straightforward way (Hansen and Jaumard 1995). But (4) is a difficult optimization problem, when the considered search space is multidimensional ($n \geq 2$).

Apart from some methods such, as a cyclic coordinate-wise optimization (Piyavskii 1972) and space-filling Peano curve techniques (Strongin 1992), that reduce the multivariate to the univariate case for a rectangular feasible set, convergent deterministic methods for solving the multivariate unconstrained problem fall into 2 main classes.

The first class contains direct extensions of Piyavskii's method (Piyavskii 1972) to the multivariate case and various modifications with different norms or close approximations (Mladineo 1986, 1992; Meewella and Mayne 1988; Mayne and Polak 1984; Horst and Tuy 1987; Wood 1991). Note that when the Euclidean norm is used in the multivariate case, the upper bounding functions are envelopes of circular cones with parallel symmetry axes. A problem of finding the maximum of such a bounding function becomes a difficult global optimization problem involving a system of quadratic and linear equations. Most of these approaches are quite ingenious from a theoretical viewpoint, but the inherent difficulty of sub-problems has limited practical applicability to dimension $n = 2$, in general unconstrained problems. For an excellent survey and numerical tests, see (Hansen and Jaumard 1995). Most of these algorithms can be improved by interpreting them as branch-and-bound methods (Hansen and Jaumard 1995; Horst and Tuy 1988, 1993).

The second class contains many simplicial and rectangular branch-and-bound techniques, but, in general, considerably weaker bounds (Galperin 1985; Horst 1988; Pinter 1986a, 1988; Tuy and Horst 1988). They differ in the ways how branching is performed and bounds are computed. Simplicial partitions are preferable when the values of an objective function at the vertices of partitions are used to compute bounds (Žilinskas, A. and Žilinskas, J. 2002; Žilinskas 2008). Another advantage of simplicial partitions is that they may be used to vertex-triangulate feasible regions of a non-rectangular shape defined by linear inequality constraints (Žilinskas 2008), what allows reduction of search space of problems with symmetric objective functions (Žilinskas 2007). In general, bounds belong to the following 2 simple families $\mu_1(P)$ and $\mu_2(P)$.

Let

$$\delta(P) = \max\left\{\|x-y\| : x, y \in P\right\}$$

denote the diameter of P. For example, if $P = \left\{x \in \Re^n : a \le x \le b\right\}$ is an n-rectangle, then $\delta(P) = \|b-a\|$, and if P is an n-simplex, then the diameter $\delta(P)$ is the length of its longest edge. Afterwards a simpler upper bound can be derived from (3):

$$\mu_1(P) = \min_{y \in T} f(y) + L\delta(P) \tag{5}$$

where $T \subset P$ is a finite sample of points in P, where the function values of f have been evaluated. If P is a rectangle or a simplex, the set T often coincides with the vertex set $V(P)$. A more tight but computationally more expensive than (5) bound is

$$\mu_2(P) = \min_{y \in T}\left\{ f(y) + L \max_{z \in V(P)} \|y-z\| \right\}. \tag{6}$$

Methods from both classes have been tested on certain problems. Methods with Piyavskii's bound (4), computed exactly or within a tight tolerance, can hardly be used to solve typical test problems with relatively large Lipschitz constants and $n > 2$. The best algorithms of the second class can almost always provide reasonable approximate optimal solutions for $n = 3$. A number of interesting practical problems can be solved by the methods of the second class up to $n = 5$ (Hendrix and Pinter 1991; Wood 1991; Pinter 1986b). For $n = 2$ the methods of the second class usually involve more function evaluations (and thus are less suitable in case of very expensive functions), but much less computational time than the methods of the first class.

In this work, we propose an improved combination of bounds of both classes. It has been suggested in (Žilinskas 2000) to estimate the bounds for the optimum over the simplex using function values at one or more vertices. The lower bound for the optimum is the largest value of the function at the vertex:

$$LB(I) = \max_{v \in V(I)} f(v),$$

where v is a vertex of the simplex I and $V(I)$ – a vertex set. In (Paulavičius and Žilinskas 2006, 2007) the combination of bounds, based on the extreme (infinite and first) and Euclidean norms over the multidimensional simplex I, was proposed:

$$UB(I) = \min_{v \in V(I)} \left\{ f(v) + K \right\}, \tag{7}$$

where

$$K = \min\{L_1 \max_{x \in I} \|x - v\|_\infty, L_2 \max_{x \in I} \|x - v\|_2, L_\infty \max_{x \in I} \|x - v\|_1\}.$$

An improved bound based on the first norm was proposed in (Paulavičius and Žilinskas 2008b), where a method from the first class is used:

$$F(I) = \max_{x \in I} \left(\min_{x_v \in I} \left\{ f(x_v) + L_\infty \|x - x_v\|_1 \right\} \right). \tag{8}$$

In this case, the upper bounding function is the envelope of n-dimensional pyramids (Fig. 1) and its maximum point is found by solving a system of linear equations. In the case of the Euclidean norm, the upper bounding function is the envelope of n-dimensional cones (Fig. 2) and its maximum point is found by solving a system of quadratic and linear equations. Therefore, the bound based on the first norm is less computationally expensive.

3. Branch-and-bound with a combination of various bounds for Lipschitz optimization

A branch-and-bound technique may be used for implementing the covering global optimization methods (Žilinskas 2008; Paulavičius and Žilinskas 2009) as well as combinatorial optimization algorithms (Žilinskas, A. and Žilinskas, J. 2009). Branch-and-bound algorithms divide a feasible region into sub-regions and detect sub-regions that cannot contain the global optimizer, by evaluating bounds for the optimum over the considered sub-regions. Performance of branch-and-bound algorithms depends on tightness of bounds (Žilinskas, A. and Žilinskas, J. 2006). Bounds may be estimated using the interval arithmetic (Žilinskas 2005, 2006) or its modifications (Žilinskas and Bogle 2003, 2004, 2007, 2009) as well as the Lipschitz condition. The optimization stops when global optimizers are bracketed in small sub-regions guaranteeing the required accuracy.

A general n-dimensional simplex-based branch-and-bound algorithm for Lipschitz optimization has been proposed in (Žilinskas 2000). We use a modification of the algorithm with various bounds. The rules of selection, covering, branching, and bounding have been corroborated by the results of experimental investigations. Simplicial partitions are used because they are preferable, when the values of an objective function at the vertices of partitions are used to compute bounds.

The feasible region should be initially covered by simplices for simplex-based branch-and-bound. The experiments in (Žilinskas 2000) have shown that the most preferable initial

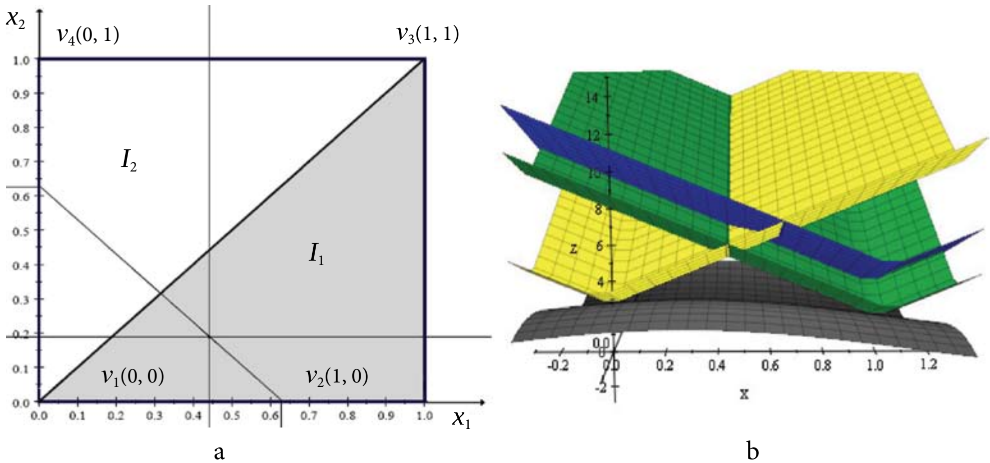

Fig. 1. Projection of intersection lines (a); visualization of upper bounding functions with the first norm (b)

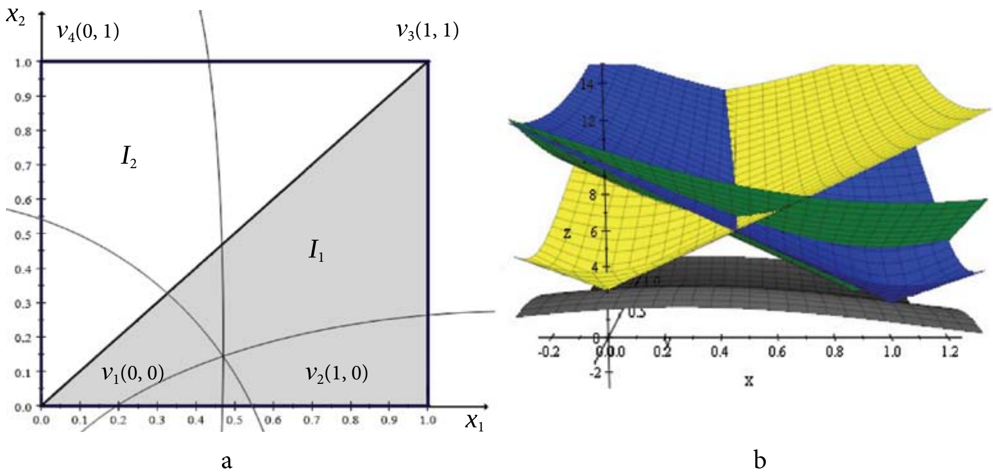

Fig. 2. Projection of intersection curves (a); visualization of upper bounding functions with Euclidean norm (b)

covering is face-to-face vertex triangulation – partitioning of the feasible region into finitely many n-dimensional simplices, whose vertices are also the vertices of the feasible region. The general (any dimensional) algorithm for combinatorial vertex triangulation of a hyper-rectangle is given in (Žilinskas 2008). The algorithm constructs initial simplicial partitioning from the bounds of variables that define the hyper-rectangular feasible region. The approach is deterministic, the number of simplices is known in advance, it is equal to $n!$. All simplices are of equal hyper-volume.

There are several ways to divide a simplex into sub-simplices. The experiments in (Žilinskas 2000) have shown that the most preferable partitioning is subdivision of a simplex into 2 by a hyper-plane passing through the middle point of the longest edge and the vertices which do not belong to the longest edge.

The branch-and-bound process is illustrated using a simple example in Figs 3 and 4. In this example $n = 1$, D = [–2,2], $f(x)$ is a Lipschitz function. As the example is one dimensional, initial covering is trivial: the feasible region is the initial simplex. Then it is divided into 2 through the middle point (first step) and both are subdivided into 2 (second step) again. At this step of the search there are 4 simplices $I_1, \ldots, I_4$ (Fig. 3). Two different upper bounds are shown (UB_i(7) and F_i(8)) for each simplex. All the norms are equal in the case $n = 1$, therefore the bound is

$$UB(I) = \min_{x_v \in I} \left\{ f(x_v) + L \max_{x \in I} \|x - x_v\| \right\}.$$

For all simplices $I_i : UB_i > LB$, but for I_4 the improved bound $F_4 < LB$. Therefore I_4 can be discarded from the further search, if the improved bound is used. Non-discarded simplices are subdivided further in the third step (Fig. 4). There are 6 simplices in the case of improved bounds and 8 simplices in the case of bound (7). Again, 2 different upper bounds are shown (UB_i (7) and F_i (8)) for each simplex $UB_i < LB$ for $i = 4, 5, 6, 7$ and $F_j < LB$ for $j = 3, 4, 5, 6$. Two simplices remained in the case of the improved bound and 5 in the case of (7). Therefore, using the improved bound, we can faster detect sub-regions that cannot contain the global optimizer.

However, the bounds based on the first norm are not always best (Paulavičius and Žilinskas 2008b). In some cases, combinations of bounds (7) may give better results. Therefore the combination of (7) and (8) (Paulavičius and Žilinskas 2008a) is used in this work:

$$\mu(I) = \min\{UB(I), F(I)\} = \min_{v \in V(I)} \{ f(v) + \min K' \}, \tag{9}$$

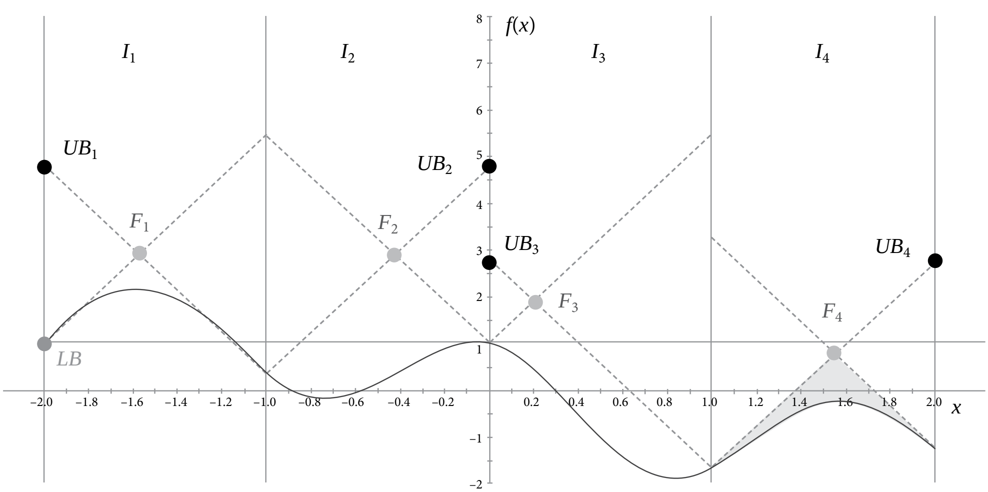

Fig. 3. Second step of branch-and-bound with simplicial partitions with $n = 1$

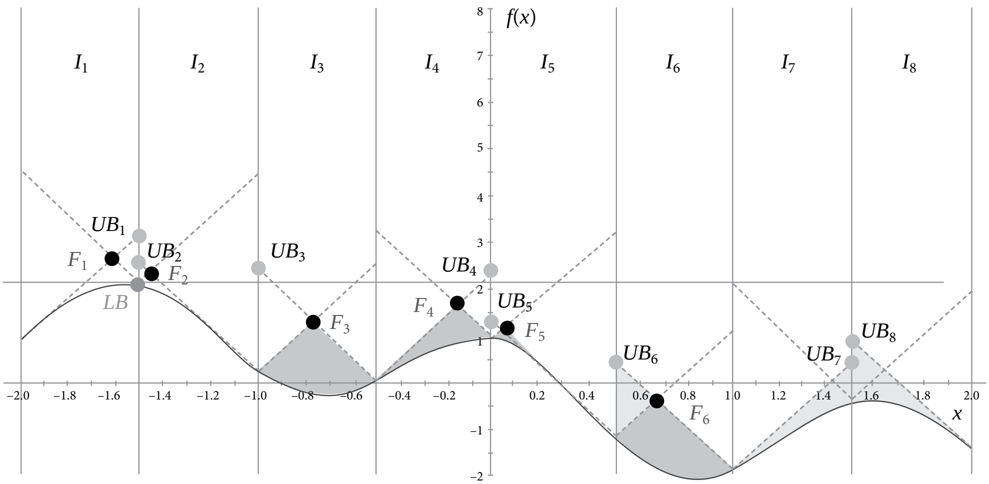

Fig. 4. Third step of branch-and-bound with simplicial partitions as $n = 1$

where

$$K' = \left\{ L_1 \max_{x \in I} \left\| x - x_v \right\|_\infty, L_2 \max_{x \in I} \left\| x - x_v \right\|_2, \max_{x \in I} \left(\min_{x_v \in I} \left\{ f(x_v) + L_\infty \left\| x - x_v \right\|_1 \right\} \right) \right\}.$$

The branch-and-bound algorithm with a combination of bounds is shown in Algorithm 1.

Algorithm 1. Branch-and-bound algorithm with combination of bounds

An n-dimensional hyper-rectangle D is face-to-face vertex triangulated into a set of n-dimensional simplices $I = \{ I_k \mid D \subseteq \cup I_k,\ k = 1, \ldots, n! \}$

$LB(D) = -\infty$

while (I is not empty: $I \neq \varnothing$) **do**

Choose and exclude $I_k \in I$ from the set of non-solved simplices I.

$LB(D) = \max\left(LB(D),\ \max_{v \in V(I)} f(v) \right)$

$UB(I_k) = \min\{ UB(I_k), F(I_k) \}$

if $\left(UB(I_k) - LB(D) > \varepsilon \right)$ **then**

Subdivide I_k into 2 simplices: $I_{k1},\ I_{k2}$

$I = I \cup \{ I_{k1},\ I_{k2} \}$

end if

end while

4. Results of experiments

The purpose of this section is to compare the proposed branch-and-bound algorithm for global optimization with other well-known algorithms for Lipschitz optimization. Various test problems for global optimization from (Hansen, Jaumard 1995; Jansson and Knüppel 1992; Madsen and Žilinskas 2000) have been used in our experiments. Test functions with $n = 2$ and $n = 3$ are numbered according to (Hansen and Jaumard 1995) and (Madsen and Žilinskas 2000). The names of functions from (Jansson and Knüppel 1992) are used for the case of higher dimensionality $(n \geq 4)$. The speed of global optimization has been estimated using the criterion of the number of function evaluations.

4.1. Computational comparison of various bounds

In (Paulavičius and Žilinskas 2006), we have shown that, for dimension $n = 2$, a combination of bounds based on 2 extreme (infinite and first) norms gives by 22% smaller number of function evaluations than the bound based on the Euclidean norm, and, for dimension $n = 3$, combination (7) gives 39% smaller number of function evaluations than in the case the Euclidean norm is used alone. In (Paulavičius and Žilinskas 2007), we have shown that, for $n = 4$, the number of function evaluations is smaller by 13% on the average and, for $n = 5, 6$, the number of function evaluations is smaller by 37% on the average, if combination (7) is used. In (Paulavičius and Žilinskas 2008b) it has been shown that the improved upper bound F (8) gives better results for Lipschitz optimization than that used as usual. Depending on the dimensionality of test problems, the number of function evaluations is from 4% to 30% smaller than with a simpler bound.

The further investigation has shown (Table 1) that a combination of the improved upper bound based on the first norm F (8) and simple bounds based on different norms UB (7) yield better results. The number of function evaluations is up to 20 times smaller when the proposed combination (9) is used than using the upper bound (8) and up to 1.6 times smaller than using a combination of simple bounds based on different norms (7). On average the numbers of function evaluations are smaller by 14%, if (9) is used as compared to (7), and by 36% smaller compared to (8).

4.2. Computational comparison of two- and three-dimensional test functions with other algorithms

In this section, the results of the proposed algorithm are compared with a representative series of algorithms for a multivariate Lipschitz optimization described in (Hansen and Jaumard 1995). Two classes of algorithms are considered:

(I) Algorithms using a single upper-bounding function, i.e. variants of Piyavskii's algorithm (Piyavskii 1972):

– Mladineo (*MLA86*) (Mladineo 1986);
– Jaumard, Herrmann and Ribault (*JHR*) (Hansen and Jaumard 1995);
– Wood (*WOOD*) (Wood 1991).

(II) Branch-and-bound algorithms:

– With the combination μ (9) proposed by us.
– Galperin (*GAL85*, *GAL88*) (Galperin 1985, 1988);

Table 1. The numbers of function evaluations for algorithm with various bounds

The name of test problem	n	ε	μ (9)	F(8)	UB (7)
1. Hansen and Jaumard 1995	2	0.355	553	1085	556
2. Hansen and Jaumard 1995	2	0.0446	93	181	158
3. Hansen and Jaumard 1995	2	11.9	3689	5259	4168
4. Hansen and Jaumard 1995	2	0.0141	9	30	9
5. Hansen and Jaumard 1995	2	0.1	40	59	62
6. Hansen and Jaumard 1995	2	44.9	1014	1552	1038
7. Hansen and Jaumard 1995	2	542.0	9172	15 180	10 683
8. Hansen and Jaumard 1995	2	3.66	249	287	314
9. Hansen and Jaumard 1995	2	62900	19 977	33 871	20 672
10. Hansen and Jaumard 1995	2	0.691	1056	1209	1285
11. Hansen and Jaumard 1995	2	0.335	2742	3290	3085
12. Hansen and Jaumard 1995	2	0.804	11 784	13 555	14 929
13. Hansen and Jaumard 1995	2	6.92	11 531	21 672	11 724
20. Hansen and Jaumard 1995	3	2.12	805 663	902 342	901 737
21. Hansen and Jaumard 1995	3	0.369	1095	12 235	1097
23. Hansen and Jaumard 1995	3	8.33	536 845	>3 000 000	536 846
24. Hansen and Jaumard 1995	3	0.672	336 241	354 488	353 191
25. Hansen and Jaumard 1995	3	0.0506	4924	10 477	5107
26. Hansen and Jaumard 1995	3	4.51	6165	6387	6589
Rosenbrock (Madsen and Žilinskas 2000)	3	2500.0	122 382	170 363	124 615
Levy No. 15 (Jansson and Knüppel 1992)	4	$0.5L_2$	>3 000 000	>3 000 000	>3 000 000
Rosenbrock (Madsen and Žilinskas 2000)	4	$0.5L_2$	453 502	473 706	467 706
Shekel 5 (Jansson and Knüppel 1992)	4	$0.5L_2$	2 048 605	2 155 904	2 355 895
Shekel 7 (Jansson and Knüppel 1992)	4	$0.5L_2$	2 047 985	2 145 884	2 345 221
Shekel 10 (Jansson and Knüppel 1992)	4	$0.5L_2$	2 048 843	2 155 204	2 355 195
Schwefel 1.2 (Jansson and Knüppel 1992)	4	$0.5L_2$	1 644 240	>3 000 000	1 734 839
Powell (Jansson and Knüppel 1992)	4	$0.5L_2$	209 498	573 025	213 055
Levy No. 9 (Jansson and Knüppel 1992)	4	$0.5L_2$	522 863	530 065	547 180
Levy No. 16 (Jansson and Knüppel 1992)	5	$1.5L_2$	137 163	>3 000 000	137 169
Rosenbrock (Madsen and Žilinskas 2000)	5	$1.5L_2$	553 373	572 647	558 423
Levy No. 10 (Jansson and Knüppel 1992)	5	1.5L2	710 156	7 125 470	721 940
Schwefel 3.7 (Jansson and Knüppel 1992)	5	$1.5L_2$	32	32	32
Levy No. 10 (Jansson and Knüppel 1992)	6	$3L_2$	103 780	110 648	137 159
Rosenbrock (Madsen and Žilinskas 2000)	6	$3L_2$	351 443	383 435	558 438

– Pinter (*PINTER*) (Pinter 1986c);
– Meewella and Mayne (*MM*) (Meewella and Mayne 1988);
– Gourdin, Hansen and Joumard (*GHJ*) (Gourdin *et al.* 1994).

The comparison of algorithms is based on the number of function evaluation criteria. The numbers of function evaluations are presented in Tables 2, 3, 4, and 5. The efficiency (Table 5) is defined as the number of function evaluations, used by our algorithm, divided by the number of function evaluations, used by other algorithms. μ (9) represents the experimental computational performance of the proposed algorithm, while the performance of the other algorithms is taken from (Hansen and Jaumard 1995). It is mentioned in (Hansen and Jaumard 1995) that the results for all algorithms may be obtained only when the required precision is not too restrictive. Even so, some problems cannot be solved by some algorithms in reasonable computational time and/or memory size. In the experiments we apply the precision used in (Hansen and Jaumard 1995). The numbers of function evaluations are smallest, when the algorithms of Mladineo and of Jaumard, Herrmann and Ribault are used. However, these algorithms belong to the first class and require a longer computational time. The branch-and-bound algorithms require larger numbers of function evaluations, but much shorter computational time. The performance of the proposed algorithm is similar to that of the best branch-and-bound algorithm (*GHJ*) and often it is even better.

5. Conclusions

In this paper, an improved general algorithm for multidimensional Lipschitz global optimization is proposed and tested. Test problems of various dimensionalities ($n = 2, 3, 4, 5, 6$) from the literature have been used for experimental investigation of the algorithm. The proposed branch-and-bound algorithm with a combination of Lipschitz bounds requires less function evaluations than the algorithms with simpler bounds. The numbers of function evaluations are 36% smaller than that using the improved upper bound based on the first norm alone and 14% smaller than using a combination of simple bounds.

The results of the proposed algorithm are compared with the performance of other algorithms for Lipschitz optimization. The performance of the proposed algorithm is similar to that of the best branch-and-bound algorithm for Lipschitz optimization and it is often better.

Investigation and application of Lipschitz global optimization would be desirable to estimate parameter values of the models of markets in disequilibrium and therefore it is a direction of further research.

Acknowledgements

The research is supported by Lithuanian State Science and Studies Foundation within the project B-03/2007 "Global optimization of complex systems using high performance computing and grid technologies".

Table 2. The numbers of function evaluations for $n = 2$

Test problem	ε	MLA86	JHR	WOOD	μ (9)	GAL85	GAL88	PINTER	MM	GHJ
1	0.355	320	323	5528	553	3553	1713	3807	1749	643
2	0.044 6	80	80	2861	93	1036	577	1762	744	167
3	11.9	2066	2066	70 955	3689	24 214	16 089	28 417	10 839	3531
4	0.0141	6	6	157	9	106	73	1527	94	45
5	0.1	41	41	209	40	430	217	907	424	73
6	44.9	548	548	14 740	1014	7729	2929	7772	2684	969
7	542.0	-	5088	183 759	9172	43 123	34 705	62 917	22 799	7969
8	3.66	177	177	1403	249	2113	1289	2272	964	301
9	62 900	–	8838	309 763	19 977	57 814	49 873	88 932	53 549	13 953
10	0.691	673	673	18 613	1056	8508	5628	9022	3814	1123
11	0.335	1613	1613	53 348	2742	18 235	12 737	20 312	9224	2677
12	0.804	–	8414	470 200	11 784	63 088	56 177	105 572	45 389	12 643
13	6.92	–	9617	–	11 531	65 536	59 049	109 227	35 949	15 695

Table 3. The numbers of function evaluations for $n = 2$ with higher precision ε

Test problem	ε	MLA86	JHR	μ(9)	GAL85	GAL88	PINTER	GHJ
1	0.035 5	913	915	1456	8713	4513	12 412	1711
2	0.004 46	342	342	465	3628	2065	8207	621
3	1.19	–	28 047	50 539	261 322	192 785	358 937	45 557
4	0.001 41	10	10	9	151	105	2452	69
5	0.01	81	81	69	781	417	1972	163
6	4.49	–	2099	4219	26 215	11 449	29 387	3555
7	54.20	–	168 325	312 967	1 153 060	1 047 617	>400 000	293 337
8	0.366	1335	1335	1574	13117	8617	16937	2271
9	6290.0	–	–	932 331	2 598 898	2 661 929	>400 000	526 253
10	0.069 1	–	6787	10 448	66 423	47 236	85 052	11 185
11	0.033 5	–	24 192	42 246	224 908	170 289	313 217	41 743
12	0.080 4	–	–	274 156	2 090 938	1 706 705	>400 000	378 759
13	0.692	–	–	323 158	2 678 542	1 620 545	>400 000	388 325

Table 4. The numbers of function evaluations for $n = 3$

Test problem	ε	MLA86	JHR	μ (9)	GAL85	GAL88	GHJ
20	2.12	>460	>41 700	805 663	5 383 113	3 886 897	215 061
21	0.369	>290	9363	1095	635 909	347 075	24 249
23	8.33	>290	>12 000	536 845	1 562 0627	–	1 297 205
24	0.672	>280	>14 400	336 241	1 248 1708	–	268 279
25	0.0506	>690	1309	4924	46 411	23 765	3219
26	4.51	446	445	6165	35 463	18 669	7177

Table 5. Efficiency for $n = 2, 3$ test problems

Test problem	MLA86	JHR	WOOD	$\mu(9)$	GAL85	GAL88	PINTER	MM	GHJ
1.	1.73	1.71	0.10	1.00	0.16	0.32	0.15	0.32	0.86
2.	1.16	1.16	0.03	1.00	0.09	0.16	0.05	0.13	0.56
3.	1.79	1.79	0.05	1.00	0.15	0.23	0.13	0.34	1.04
4.	1.50	1.50	0.06	1.00	0.08	0.12	0.01	0.10	0.20
5.	0.98	0.98	0.19	1.00	0.09	0.18	0.04	0.09	0.55
6.	1.85	1.85	0.07	1.00	0.13	0.35	0.13	0.38	1.05
7.	–	1.80	0.05	1.00	0.21	0.26	0.15	0.40	1.15
8.	1.41	1.41	0.18	1.00	0.12	0.19	0.11	0.26	0.83
9.	–	2.26	0.06	1.00	0.35	0.40	0.22	0.37	1.43
10.	1.57	1.57	0.06	1.00	0.12	0.19	0.12	0.28	0.94
11.	1.70	1.70	0.05	1.00	0.15	0.22	0.13	0.30	1.02
12.	–	1.40	0.03	1.00	0.19	0.21	0.11	0.26	0.93
13.	–	1.20	–	1.00	0.18	0.20	0.11	0.32	0.73
20	–	–		1.00	0.15	0.21			3.75
21	–	0.12		1.00	0.00	0.00			0.05
23	–	–		1.00	0.03	–			0.41
24	–	–		1.00	0.03	–			1.25
25	–	3.76		1.00	0.11	0.21			1.53
26	13.82	13.85		1.00	0.17	0.33			0.86

References

Dorsey, R. E.; Mayer, W. J. 1995. Genetic algorithms for estimation problems with multiple optima, nondifferentiability, and other irregular features, *Journal of Business & Economic Statistics* 13(1): 53–66.

Fair, R. C. 1971. *A Short-Run Forecasting Model of the United States Economy.* Heath Lexington Books.

Galperin, E. A. 1985. The cubic algorithm, *Journal of Mathematical Analysis and Applications* 112: 635–640.

Galperin, E. A. 1988. Precision, complexity, and computational schemes of the cubic algorithm, *Journal of Optimization Theory and Applications* 57: 223–238.

Gourdin, E.; Hansen, P.; Jaumard, B. 1994. *Global optimization of multivariate Lipschitz functions: survey and computational comparison. Les Cahiers du GERAD.*

Hansen, P.; Jaumard, B. 1995. Lipshitz optimization, in Horst, R. and Pardalos, P. M. (Eds.). *Handbook of Global Optimization*. Kluwer Academic Publishers, 404–493.

Hendrix, E. M. T.; Pinter, J. 1991. An application of Lipschitzian global optimization to product design, *Journal of Global Optimization* 2: 389–402.

Horst, R. 1988. Optimization with partition sets whose feasibility is not known. Application to concave minimization, reverse convex constraints, D. C.-programming and Lipschitzian optimization, *Journal of Optimization Theory and Application* 58: 11–37.

Horst, R.; Tuy, R. 1987. On the convergence of global methods in multiextremal optimization, *Journal of Optimization Theory and Applications* 54: 253–271.

Horst, R.; Tuy, H. 1988. Branch-and-bound methods for solving systems of Lipschitzian equations and inequalities, *Journal of Optimization Theory and Applications* 58: 139–146.

Horst, R.; Tuy, H. 1993. *Global optimization: deterministic approaches*. Springer, 2 edition.

Jansson, C.; Knüppel, O. A. 1992. *Global Minimization Method: The Multi-Dimensional Case.* Technical Report TU Hamburg-Harburg.

Jerrell, M. E.; Campione, W. A. 2001. Global optimization of econometric functions, *Journal of Global Optimization* 20(3–4): 273–295.

Maddala, G. S. 1983. *Limited-Dependent and Qualitative Variables in Econometrics*. Cambridge University Press.

Maddala, G. S.; Nelson, F. D. 1974. Maximum likelihood methods for models of markets in disequilibrium, *Econometrica* 42(6): 1013–1030.

Madsen, K.; Žilinskas, J. 2000. *Testing Branch-and-Bound Methods for Global Optimization*. Technical Report IMM-REP-2000-05 Technical University of Denmark.

Mayne, D. Q.; Polak, E. 1984. Outer approximation algorithm for nondifferentiable optimization problems, *Journal of Optimization Theory and Applications* 42(1): 19–30.

Meewella, C. C.; Mayne, D. Q. 1988. An algorithm for global optimization of Lipshitz continuous functions, *Journal of Optimization Theory and Applications* 57: 307–323.

Mladineo, R. H. 1986. An algorithm for finding the global maximum of a multimodal, multivariate function, *Mathematical Programming* 34: 188–200.

Mladineo, R. H. 1992. Convergence rates of a global optimization algorithms, *Mathematical Programming* 54: 223–232.

Paulavičius, R.; Žilinskas, J. 2006. Analysis of different norms and corresponding Lipschitz constants for global optimization, *Technological and Economic Development of Economy* 12(4): 301–306.

Paulavičius, R.; Žilinskas, J. 2007. Analysis of different norms and corresponding Lipschitz constants for global optimization in multidimensional case, *Information Technology and Control* 36(4): 383–387.

Paulavičius, R.; Žilinskas, J. 2008a. Branch and bound with simplicial partitions and combination of Lipschitz bounds for global optimization, in Sakalauskas, L.; Weber, G. W. and Zavadskas, E. K. (Eds.). *20th International Conference EURO Mini Conference on Continuous Optimization and Knowledge-Based Technologies (EurOPT-2008), May 20-23, 2008, Neringa, Lithuania*. Vilnius: Technika, 54–58.

Paulavičius, R.; Žilinskas, J. 2008b. Improved Lipschitz bounds with first norm for function values over multidimensional simplex, *Mathematical Modelling and Analysis* 13(4): 553–563.

Paulavičius, R.; Žilinskas, J. 2009. Parallel branch and bound algorithm with combination of Lipschitz bounds over multidimensional simplices for multicore computers, in Čiegis, R.; Henty, D.; Kågström, B. and Žilinskas, J. (Eds.). *Parallel Scientific Computing and Optimization*, Vol. 27 of Springer Optimization and Its Applications. Springer, 93–102.

Pinter, J. 1986a. Extended univariate algorithms for *n*-dimensional global optimization, *Computing* 36: 91–103.

Pinter, J. 1986b. Global optimization on convex sets, *Operations Research Spektrum* 8: 197–202.

Pinter, J. 1986c. Globally convergent methods for *n*-dimensional multiextremal optimization, *Optimization* 17: 187–202.

Pinter, J. 1988. Branch-and-bound algorithms for solving global optimization problems with Lipschitzian structure, *Optimization* 19: 101–110.

Piyavskii, S. A.1972. An algorithm for finding the absolute extremum of a function, *Zh. Vychisl. Mat. mat. Fiz* 12(4): 888–896.

Strongin, R. G. 1992. Algorithms for multiextremal mathematical programming problems employing the set of joint space-filling curves, *Journal of Global Optimization* 2: 357–378.

Tuy, H.; Horst, R. 1988. Convergence and restart in branch and bound algorithms for global optimization. Application to concave minimization and DC-optimization problems, *Mathematical Programming* 41: 161–183.

Wood, G. R. 1991. Multidimensional bisection applied to global optimization, *Computers &Mathematics with Applications* 21: 161–172.

Žilinskas, A.; Žilinskas, J. 2002. Global optimization based on a statistical model and simplicial partitioning, *Computers & Mathematics with Applications* 44(7): 957–967.

Žilinskas, A.; Žilinskas, J. 2006. On efficiency of tightening bounds in interval global optimization, *Lecture Notes in Computer Science* 3732: 197–205.

Žilinskas, A.; Žilinskas, J. 2009. Branch and bound algorithm for multidimensional scaling with city-block metric, *Journal of Global Optimization* 43(2–3): 357–372.

Žilinskas, J. 2000. Optimization of Lipschitzian functions by simplex-based branch and bound, *Information Technology and Control* 14(1): 45–50.

Žilinskas, J. 2005. Comparison of packages for interval arithmetic, *Informatica* 16(1): 145–154.

Žilinskas, J. 2006. Estimation of functional ranges using standard and inner interval arithmetic, *Informatica* 17(1): 125–136.

Žilinskas, J. 2007. Reducing of search space of multidimensional scaling problems with data exposing symmetries, *Information Technology and Control* 36(4): 377–382.

Žilinskas, J. 2008. Branch and bound with simplicial partitions for global optimization, *Mathematical Modelling and Analysis* 13(1): 145–159.

Žilinskas, J.; Bogle, I. D. L. 2003. Evaluation ranges of functions using balanced random interval arithmetic, *Informatica* 14(3): 403–416.

Žilinskas, J.; Bogle, I. D. L. 2004. Balanced random interval arithmetic, *Computers & Chemical Engineering* 28(5): 839–851.

Žilinskas, J.; Bogle, I. D. L. 2006. Balanced random interval arithmetic in market model estimation, *European Journal of Operational Research* 175(3): 1367–1378.

Žilinskas, J.; Bogle, I. D. L. 2007. A survey of methods for the estimation ranges of functions using interval arithmetic, in Törn, A. and Žilinskas, J. (Eds.). *Models and Algorithms for Global Optimization*, Vol. 4 of Springer Optimization and Its Applications. Springer, 97–108.

Žilinskas, J.; Bogle, I. D. L. 2009. Global optimization: Interval analysis and balanced interval arithmetic, in Floudas, C. A. and Pardalos, P. M. (Eds.). *Encyclopedia of Optimization*, 2nd edn., Springer, 1346–1350.

GLOBALUSIS OPTIMIZAVIMAS ŠAKŲ IR RĖŽIŲ ALGORITMU SU LIPŠICO RĖŽIŲ JUNGINIU SIMPLEKSE

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Santrauka

Daug įvairių ekonomikos uždavinių yra formuluojami kaip globaliojo optimizavimo uždaviniai. Didžioji dalis Lipšico globaliojo optimizavimo metodų, tinkamų spręsti didesnės dimensijos, t. y. $n > 2$, uždavinius, naudoja stačiakampį arba simpleksinį šakų ir rėžių metodus bei paprastesnius rėžius. Šiame darbe pasiūlytas simpleksinis šakų ir rėžių algoritmas, naudojantis dviejų tipų viršutinių rėžių junginį. Pirmasis yra pagerintas rėžis su pirmąja norma, kitas – trijų paprastesnių rėžių su skirtingomis normomis junginys. Gautieji eksperimentiniai pasiūlyto algoritmo rezultatai yra palyginti su kitų gerai žinomų Lipšico optimizavimo algoritmų rezultatais.

Reikšminiai žodžiai: šakų ir rėžių algoritmas, globalusis optimizavimas, Lipšico optimizavimas, Lipšico rėžis.

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